University of Chester

Maths Notes for first year Engineers

9. Gauss Elimination

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Chapter 9

Linear Algebra

9.1 Solving Systems of Equations (Gaussian Elimination)

Here we present an algorithm for solving systems of linear equations. We can write all the information from a system of linear equations in just one matrix, called the **augmented matrix** of the linear system.

Example 9.1 *Consider the system of equations:*

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{{}+5z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

The corresponding augmented matrix to this system is $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right)$.

If we apply the following operations to the augmented matrix, we will not change the solution set:

1. Multiply a row by a non-zero constant.
2. Swap two rows.
3. Add a multiple of one row to another row.

There three operations are called **elementary row operations**. Note that if one did not like dealing with fractions, we can combine some of the elementary row operations to get:

4. Add a multiple of one row to a multiple of another row.

Example 9.2 *Use row operations to solve the following system of equations:*

$$\begin{aligned} x + 2y &= 1 \\ 2x + 3y &= 2. \end{aligned}$$

The corresponding augmented matrix is: $\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 3 & 2 \end{array}\right)$. *Now we use row operations to solve the system of equations.*

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 3 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & 0 \end{array}\right) \begin{array}{l} \\ r_2 - 2r_1 \end{array} \quad \textit{Add the 2nd row to -2 times the 1st row.}$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right) \begin{array}{l} r_1 + 2r_2 \\ \end{array} \quad \textit{Add the 1st row to 2 times the 2nd row.}$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right) \begin{array}{l} \\ r_2 \times -1 \end{array} \quad \textit{Multiply the 2nd row by } -1.$$

Therefore $1 \cdot x + 0 \cdot y = 1 \Longrightarrow x = 1$ *and* $0 \cdot x + 1 \cdot y = 0 \Longrightarrow y = 0$. *i.e.* $(x, y) = (1, 0)$.

Example 9.3 *Consider the system of equations:*

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \qquad &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

The corresponding augmented matrix to this system is $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right)$. *Now we use row operations to solve the system of equations.*

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right) \begin{array}{l} \\ r_1 + r_2 \\ r_3 - 2r_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right) \begin{array}{l} \\ \\ r_2 + r_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} \\ \\ r_3/2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} r_1 - 3r_3 \\ r_2 - 3r_3 \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} r_1 + 2r_2 \\ \\ \end{array}.$$

Therefore $(x, y, z) = (1, -1, 2)$.

Definition 9.4 *A matrix is in* ***reduced row-echelon form*** *when:*

1. *If a row is not all zeros then it's first non-zero entry is* 1 *(called a leading).*

2. *If there are any rows of zeros, they appear at the bottom of the matrix.*

3. *If any two consequetive rows do not consist entirely of zeros then the leading* 1 *of the 2nd row is to the right of the leading* 1 *in the 1st row.*

4. *Each column containing a leading* 1 *had zeros everywhere else.*

If a matrix satisfies 1., 2. 3. and 4. then we say that it is in ***row-echelon form****.*

Definition 9.5 *A system of equations is called* ***consistent*** *if it has at least one solution. Otherwise it is called* ***inconsistent****.*

Example 9.6 $\left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}\right)$ *is an augmented matrix in reduced row-echelon form. We have that* $1 \cdot x = 0 \Longrightarrow x = 0$, $0 \cdot x = 1 \Longrightarrow 0 = 1$, *which is a contradiction, therefore this system has no solution (i.e. it is inconsistent).*

Example 9.7 $\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$ *is an augmented matrix in reduced row-echelon form.*

Example 9.8 $\left(\begin{array}{ccccc|c} 1 & 2 & 5 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$ *is an augmented matrix in row-echelon form.*

Definition 9.9 *A variable is called a* ***leading variable*** *if it corresponds to a leading* 1 *in a row-echelon form of the augmented matrix. Otherwise it is called a* ***free variable****.*

Example 9.10 *Use Gaussian elimination to solve the following system:*

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + 5x_2 + 7x_3 &= 8. \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 5 & 7 & 8 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 5 & 2 \end{array}\right) \; r_2 - 2r_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -9 & -1 \\ 0 & 1 & 5 & 2 \end{array}\right) \; r_1 - 2r_2 \; .$$

The leading variables are x_1 *and* x_2*. The free variable is* x_3*. Let* $x_3 = t$*. Therefore*

- $x_1 - 9t = -1 \Longrightarrow x_1 = -1 - 9t.$

- $x_2 + 5t = 2 \Longrightarrow x_2 = 2 - 5t.$

The solution set is $(x_1, x_2, x_3) = (-1 - 9t, 2 - 5t, t)$.

Example 9.11 *Consider the system of equations:*

$$\begin{aligned} 2y + z &= -8 \\ x - 2y - 3z &= 0 \\ -x + y + 2z &= 3 \end{aligned}$$

The corresponding augmented matrix to this system is $\left(\begin{array}{ccc|c} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{array}\right)$. *Now we use row operations to solve the system of equations.*

$$\begin{aligned}
\left(\begin{array}{ccc|c} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{array}\right) &\sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ -1 & 1 & 2 & 3 \end{array}\right) \quad \begin{array}{c} \textit{swap } r_1 \\ \textit{and } r_2 \end{array} \\
&\sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{array}\right) \quad r_3 + r_1 \\
&\sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 2 & 1 & -8 \end{array}\right) \quad \begin{array}{c} \textit{swap } r_2 \\ \textit{and } r_3 \end{array} \\
&\sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 31 \\ 0 & 0 & -1 & -2 \end{array}\right) \quad r_3 + 2r_2 \\
&\sim \left(\begin{array}{ccc|c} 1 & -2 & 0 & 6 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & -1 & -2 \end{array}\right) \quad \begin{array}{c} r_1 - 3r_3 \\ r_2 - r_3 \end{array} \\
&\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & -1 & -2 \end{array}\right) \quad r_1 - 2r_2 \\
&\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{c} r_2 \times -1 \\ r_3 \times -1 \end{array}.
\end{aligned}$$

Therefore $(x, y, z) = (-4, -5, 2)$.

Exercises

Q1 Solve the following system using Gauss Elimination:

$$x - 2y + 3z = 7$$
$$2x + y + z = 4$$
$$-3x + 2y - 2z = -10$$

Q2 Solve the following system using Gauss Elimination:

$$2x - 4y + 5z = -33$$
$$4x - y = -5$$
$$-2x + 2y - 3z = 19$$

Q3 Solve the following system using Gauss Elimination:

$$x_1 - x_2 - x_3 + 2x_4 = 0$$
$$2x_1 + x_2 - x_3 + 2x_4 = 8$$
$$x_1 - 3x_2 + 2x_3 + 7x_4 = 2$$

Q4 Solve the following system using Gauss Elimination:

$$x_1 - x_2 - x_3 + 2x_4 = 0$$
$$2x_1 + x_2 - x_3 + 2x_4 = 8$$
$$x_1 - 3x_2 + 2x_3 + 7x_4 = 2$$
$$x_1 - x_2 + x_3 - x_4 = -6$$

Answers

Q1 $x = 2, y = -1$ and $z = 1$.

Q2 $x = -\frac{1}{2}, y = 3$ and $z = -4$.

Q3 $(x_1, x_2, x_3, x_4) = (4 - 2t, 2 + t, 2 - t, t)$.

Q4 $x_1 = 0, x_2 = 4, x_3 = 0$ and $x_4 = 2$.