University of Chester

Maths Notes for first year Engineers

7. Vectors

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Contents

Chapter 7

Vector calculus

7.1 Addition, Subtraction, Scalar Multiplication, Length

Definition 7.1 *Let* $\vec{u} = u_1 i + u_2 j + u_3 k$ *and* $\vec{v} = v_1 i + v_2 j + v_3 k$ *where* $\vec{u}, \vec{v} \in \mathbb{R}^3$ *and* $u_i, v_j, \lambda \in \mathbb{R}$. *Then*

- $\vec{u} \pm \vec{v} = (u_1 \pm v_1)i + (u_2 \pm v_2)j + (u_3 \pm v_3)k.$
- $\lambda\vec{u} = (\lambda u_1)i + (\lambda u_2)j + (\lambda u_3)k \quad \forall \lambda \in \mathbb{R}.$
- *The magnitude (or length of* $\vec{u}$*) is* $|\vec{u}| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}$.

Example 7.2 *Let* $\vec{u} = 2i + 3j - k$ *and* $\vec{v} = 5i - j + 3k$ *where* $\vec{u}, \vec{v} \in \mathbb{R}^3$. *Find (i)* $\vec{u} + \vec{v}$, *(ii)* $\vec{u} - 3\vec{v}$, *(iii)* $|\vec{u}|$, *(iv)* $|\vec{v}|$.

(i) $\vec{u} + \vec{v} = 2i + 3j - k + 5i - j + 3k = 7i + 2j + 2k.$

(ii) $\vec{u} - 3\vec{v} = 2i + 3j - k - 3(5i - j + 3k) = 2i + 3j - k - 15i + 3j - 9k = -13i + 6j - 10k.$

(iii) $|\vec{u}| = \sqrt{4 + 9 + 1} = \sqrt{14}.$

(iv) $|\vec{v}| = \sqrt{25 + 1 + 9} = \sqrt{35}.$

7.2 Dot Product

Definition 7.3 *Let* $\vec{u} = u_1 i + u_2 j + u_3 k$ *and* $\vec{v} = v_1 i + v_2 j + v_3 k$ *where* $\vec{u}, \vec{v} \in \mathbb{R}^3$ *and* $u_i, v_j \in \mathbb{R}$. *Then the* ***dot product*** *of* $\vec{u}$ *and* $\vec{v}$ *is*

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{3} u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example 7.4 *Let* $\vec{u} = 2i + 3j - k$ *and* $\vec{w} = j - k$ *where* $\vec{u}, \vec{w} \in \mathbb{R}^3$. *Find* $\vec{u} \cdot \vec{w}$.

$\vec{u} \cdot \vec{w} = (2)(0) + (3)(1) + (-1)(-1) = 3 + 1 = 4.$

Theorem 7.5 (*Properties of the dot product*) *Let* $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ *and* $\lambda \in R$. *Then*

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$.

2. $(\lambda\vec{u})\cdot\vec{v} = \vec{u}\cdot(\lambda\vec{v}) = \lambda(\vec{u}\cdot\vec{v})$.

3. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$.

4. $\vec{u}\cdot\vec{u} = |\vec{u}|^2$.

5. $0\cdot\vec{u} = 0$.

Theorem 7.6 *Let $\vec{u},\vec{v}\in\mathbb{R}^3$. Then*

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$$

where θ is the angle between the vectors $\vec{u}$ and $\vec{v}$.

Example 7.7 *Find the angle between $\vec{u} = i - j - 4k$ and $\vec{v} = 2j + k$ where $\vec{u},\vec{v}\in\mathbb{R}^3$.*

$\vec{u}\cdot\vec{v} = (1)(0) + (-1)(2) + (-4)(1) = -6$, $|\vec{u}| = \sqrt{1+1+16} = \sqrt{18}$ and $|\vec{v}| = \sqrt{4+1} = \sqrt{5}$. Therefore

$$\begin{aligned} -6 &= \sqrt{18}\sqrt{5}\cos(\theta) \\ \cos(\theta) &= -\left(\frac{6}{\sqrt{18}\sqrt{5}}\right) \\ \theta &= \cos^{-1}(-0.632) \\ \theta &= 0.886 \text{ rad.} \end{aligned}$$

Theorem 7.8 *Let $\vec{u},\vec{v}\in\mathbb{R}^3$. Then $\vec{u}$ and $\vec{v}$ are orthogonal to each other if $\vec{u}\cdot\vec{v} = 0$.*

Exercises 7.2

Q1 Let $\overrightarrow{u} = 3i - j - k$, $\overrightarrow{v} = j - 5k$ and $\overrightarrow{w} = i - 4k$ where $\overrightarrow{u},\overrightarrow{v},\overrightarrow{w}\in\mathbb{R}^3$. Calculate

(i) $|\overrightarrow{u}|$, $|\overrightarrow{v}|$.

(ii) $7\overrightarrow{u} + 5\overrightarrow{v} - 3\overrightarrow{w}$, $|7\overrightarrow{u} + 5\overrightarrow{v} - 3\overrightarrow{w}|$.

(iii) $\overrightarrow{u}\cdot\overrightarrow{v}$, $\overrightarrow{v}\cdot\overrightarrow{u}$, $\overrightarrow{v}\cdot\overrightarrow{v}$, $\overrightarrow{u}\cdot\overrightarrow{w}$.

(iv) $\overrightarrow{u}\cdot(\overrightarrow{v}+\overrightarrow{w})$.

(v) Find the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$, $\overrightarrow{w}$ and $\overrightarrow{v}$.

Q2 Let $\overrightarrow{u},\overrightarrow{v},\overrightarrow{w}\in\mathbb{R}^3$. Prove that

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$.
2. $(\lambda\vec{u})\cdot\vec{v} = \vec{u}\cdot(\lambda\vec{v}) = \lambda(\vec{u}\cdot\vec{v})$.
3. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$.
4. $\vec{u}\cdot\vec{u} = |\vec{u}|^2$.
5. $0\cdot\vec{u} = 0$.

7.3 Cross Product

Definition 7.9 *Let $\vec{u} = u_1 i + u_2 j + u_3 k$ and $\vec{v} = v_1 i + v_2 j + v_3 k$ where $\vec{u}, \vec{v} \in \mathbb{R}^3$ and $u_i, v_j \in \mathbb{R}$. Then the cross product of $\vec{u}$ with $\vec{v}$ (denoted $\vec{u} \times \vec{v}$) is a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$ and is calculated as follows:*

$$\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= i \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - j \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + k \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\
&= i[u_2 v_3 - u_3 v_2] - j[u_1 v_3 - u_3 v_1] + k[u_1 v_2 - u_1 v_1].
\end{aligned}$$

Example 7.10 *Find $\vec{u} \times \vec{v}$ if $\vec{u} = i + j - 4k$ and $\vec{v} = 2i + 2j + k$ where $\vec{u}, \vec{v} \in \mathbb{R}^3$.*

$$\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} i & j & k \\ 1 & 1 & -4 \\ 2 & 2 & 1 \end{vmatrix} \\
&= i \begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} \\
&= i[(1)(1) - (-4)(2)] - j[(1)(1) - (-4)(2)] + k[(1)(2) - (1)(2)] \\
&= 9i - 9j.
\end{aligned}$$

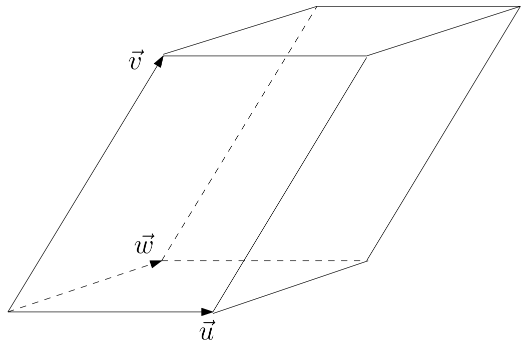
Theorem 7.11 *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $\lambda \in R$. Then*

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.
2. $(\lambda \vec{u}) \times \vec{v} = \vec{u} \times (\lambda \vec{v}) = \lambda(\vec{u} \times \vec{v})$.
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
4. $(\vec{u} + \vec{v}) \times \vec{w}) = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$.
5. $\vec{u} \times \vec{u} = 0$.
6. $\vec{u} \times 0 = 0$.

7.4 Triple Scalar Product

Definition 7.12 *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in a 3-space. Then the **triple scalar product** of $\vec{u}$, $\vec{v}$ and $\vec{w}$ is $\vec{u} \cdot (\vec{v} \times \vec{w})$.*

Theorem 7.13 *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in a 3-space. Then $\vec{u} \cdot (\vec{v} \times \vec{w})$ corresponds to the volume of a parallelepiped with adjacent sides $\vec{u}$, $\vec{v}$ and $\vec{w}$.*



The volume of a parallelepiped with adjacent sides $\vec{u}$, $\vec{v}$ and $\vec{w}$

Example 7.14 *Find the volume of the parallelepiped with adjacent sides* $\vec{u} = i + 2j + 3k$, $\vec{v} = -2i + j + 6k$ *and* $\vec{w} = 4i + 5j - k$.

$$\begin{aligned}
\vec{v} \times \overrightarrow{w} &= \begin{vmatrix} i & j & k \\ -2 & 1 & 6 \\ 4 & 5 & -1 \end{vmatrix} \\
&= i \begin{vmatrix} 1 & 6 \\ 5 & -1 \end{vmatrix} - j \begin{vmatrix} -2 & 6 \\ 4 & -1 \end{vmatrix} + k \begin{vmatrix} -2 & 1 \\ 4 & 5 \end{vmatrix} \\
&= i[-1 - 30] - j[2 - 24 + 1] + k[-10 - 4] \\
&= -31i + 22j - 14k.
\end{aligned}$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (1)(-31) + (2)(22) + (3)(-14) = -29.$$

Thus the volume of the parallelepiped with adjacent sides $\vec{u} = i + 2j + 3k$, $\vec{v} = -2i + j + 6k$ and $\vec{w} = 4i + 5j - k$ is 29 units cubed.

Theorem 7.15 *Let* $\vec{u}$, $\vec{v}$ *and* $\vec{w}$ *be vectors in a 3-space. Then* $\vec{u}$, $\vec{v}$ *and* $\vec{w}$ *are coplanar (i.e. they lie on the same plane) if* $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$.

Example 7.16 *Determine if* $\vec{a} = i + 2j + 3k$, $\vec{b} = 3i + j - k$ *and* $\vec{c} = i - k$ *are coplanar.*

$$\begin{aligned}
\vec{b} \times \vec{c} &= \begin{vmatrix} i & j & k \\ 3 & 1 & -1 \\ 1 & 0 & -1 \end{vmatrix} \\
&= i \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} - j \begin{vmatrix} 3 & -1 \\ 1 & -1 \end{vmatrix} + k \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \\
&= i[-1] - j[-3 + 1] + k[-1] \\
&= -i + 2j - k.
\end{aligned}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (1)(-1) + (2)(2) + (3)(-1) = 0.$$

Therefore $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

Exercises 7.4

Q1 Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$. Prove that

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.
2. $(\lambda\vec{u}) \times \vec{v} = \vec{u} \times (\lambda\vec{v}) = \lambda(\vec{u} \times \vec{v})$.
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
4. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$.
5. $\vec{u} \times \vec{u} = 0$.
6. $\vec{u} \times 0 = 0$.
7. $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$.
8. $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$.

Q2 Let $\overrightarrow{a} = i + 2j + 3k$, $\overrightarrow{b} = -2i + j + 6k$, $\overrightarrow{c} = 4i + 5j - k$ $\overrightarrow{d} = 3i + j - k$ and $\overrightarrow{e} = i - k$, where $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{d}, \overrightarrow{e} \in \mathbb{R}^3$. Calculate

(i) $\overrightarrow{a} \times \overrightarrow{b}$, $|\overrightarrow{c} \times \overrightarrow{e}|$

(ii) Find the volume of the parallelepiped with adjacent sides $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$.

(iii) Find the volume of the parallelepiped with adjacent sides $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{e}$.

(iv) Are $\overrightarrow{c}$, $\overrightarrow{d}$ and $\overrightarrow{e}$ coplanar?

(v) Are $\overrightarrow{a}$, $\overrightarrow{d}$ and $\overrightarrow{e}$ coplanar?

7.5 Answers

Exercises 7.3

Q1

(i) $|\overrightarrow{u}| = \sqrt{10}$, $|\overrightarrow{u}| = \sqrt{26}$.

(ii) $7\overrightarrow{u} + 5\overrightarrow{v} - 3\overrightarrow{w} = 18i - 2i - 13k$, $|7\overrightarrow{u} + 5\overrightarrow{v} - 3\overrightarrow{w}| = \sqrt{497}$.

(iii) $\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{v} \cdot \overrightarrow{u} = -1$, $\overrightarrow{v} \cdot \overrightarrow{v} = 26$, $\overrightarrow{u} \cdot \overrightarrow{w} = 3$.

(iv) $\overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) = 2$.

(v) $(\overrightarrow{u}\hat{,}\overrightarrow{v}) = 1.63\,\text{rad}$, $(\overrightarrow{w}\hat{,}\overrightarrow{v}) = 0.31\,\text{rad}$.

Exercises 7.4

Q2

(i) $\overrightarrow{a} \times \overrightarrow{b} = 9i - 12j + 5k$, $|\overrightarrow{c} \times \overrightarrow{e}| = -5i + 3j - 5k$.

(ii) $\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) = -29$.

(iii) $\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{e}) = 4$.

(iv) $\overrightarrow{c} \cdot (\overrightarrow{d} \times \overrightarrow{e}) = 7$, not coplanar.

(v) $\overrightarrow{a} \cdot (\overrightarrow{d} \times \overrightarrow{e}) = 0$, coplanar.