University of Chester

Maths Notes for first year Engineers

6. Complex Numbers

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Contents

Chapter 6

Complex Numbers

6.1 Addition, Subtraction, Multiplication, Division, Complex Conjugate and Modulus

Definition 6.1 *The set of complex numbers (denoted by* $\mathbb{C}$*) are numbers of the form*

$$\mathbb{C} = \{a + bi \,|\, a, b \in \mathbb{R},\ i = \sqrt{-1}\}.$$

Example 6.2 $2 + 3i$, $-4 - 7i$, $-5i$ *and* -11 *are examples of complex numbers.*

In this section we will define addition, subtraction, multiplication, scalar multiplcation and division of complex numbers. Also we shall define the complex conjugate and modulus of a complex number.

1. **Addition** Let $z_1 = a + bi \in \mathbb{C}$ and $z_2 = c + di \in \mathbb{C}$, then

$$z_1 + z_2 = a + bi + c + di = (a + c) + (b + d)i.$$

2. **Subtraction** Let $z_1 = a + bi \in \mathbb{C}$ and $z_2 = c + di \in \mathbb{C}$, then

$$z_1 - z_2 = a + bi - (c + di) = a + bi - c - di = (a - c) + (b - d)i.$$

3. **Scalar Multiplication** Let $z = a + bi \in \mathbb{C}$ and $\lambda \in \mathbb{R}$, then

$$\lambda z = \lambda(a + bi) = \lambda a + \lambda bi.$$

4. **Multiplication** Let $z_1 = a + bi \in \mathbb{C}$ and $z_2 = c + di \in \mathbb{C}$, then

$$\begin{aligned} z_1 z_2 &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci + bd(-1) \quad i = \sqrt{-1} \Longleftrightarrow i^2 = -1 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

Example 6.3 *Let* $z_1 = 2 + 3i$, $z_2 = -4 + 5i$ *and* $z_3 = 4 - 3i$*. Then* (*i*) $z_1 + z_2 = 2 + 3i - 4 + 5i = -2 + 8i$.

(*ii*) $2z_2 - 3z_3 = 2(-4 + 5i) - 3(4 - 3i) = -8 + 10i - 12 + 9i = -20 + 21i$.

(*iii*) $z_1 z_2 = (2 + 3i)(-4 + 5i) = -8 + 10i - 12i + 15i^2 = -8 + 10i - 12i - 15 = -23 - 2i$.

Before we define division of complex number, we shall define the complex conjugate of a complex number.

Definition 6.4 *The complex conjugate conjugate (denoted $\overline{z}$) of a complex number $z = a + bi$ is*

$$\overline{z} = a - bi.$$

Example 6.5 *Let $z_1 = 2 + 3i$ and $z_2 = -4 - 5i$. Then*

(*i*) $\overline{z_1} = 2 - 3i$.

(*ii*) $\overline{z_2} = -4 + 5i$.

Consider $z\overline{z}$.

$$\begin{aligned} z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 - abi + abi - b^2 i^2 \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2. \end{aligned}$$

We conclude that when we multiply z by $\overline{z}$ (or $\overline{z}$ by z), the imaginary part disappears. At this point let's define division of complex numbers.

5. **Division** Let $z_1 = a + bi \in \mathbb{C}$ and $z_2 = c + di \in \mathbb{C}$. Consider $\dfrac{z_1}{z_2}$. Our problem is that we have to get rid of the complex number on the bottom. As discussed earlier, to get rid of the imaginary part of a complex number we multiply the complex number by its complex conjugate (i.e. multiply the bottom by $\overline{z_2}$). In this instance we can't just multiply the bottom by $\overline{z_2}$, we also have to multiply the top by $\overline{z_2}$. i.e.

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{z_1}{z_2}\frac{\overline{z_2}}{\overline{z_2}} \\ &= \frac{(a + bi)}{(c + di)}\frac{(c - di)}{(c - di)} \\ &= \frac{ac + adi + bci + bdi^2}{c^2 + d^2} \\ &= \frac{ac + adi + bci - bd}{c^2 + d^2} \\ &= \frac{(ac - bd) + (ad + bc)i}{c^2 + d^2} \\ &= \left(\frac{ac - bd}{c^2 + d^2}\right) + \left(\frac{ad + bc}{c^2 + d^2}\right) i. \end{aligned}$$

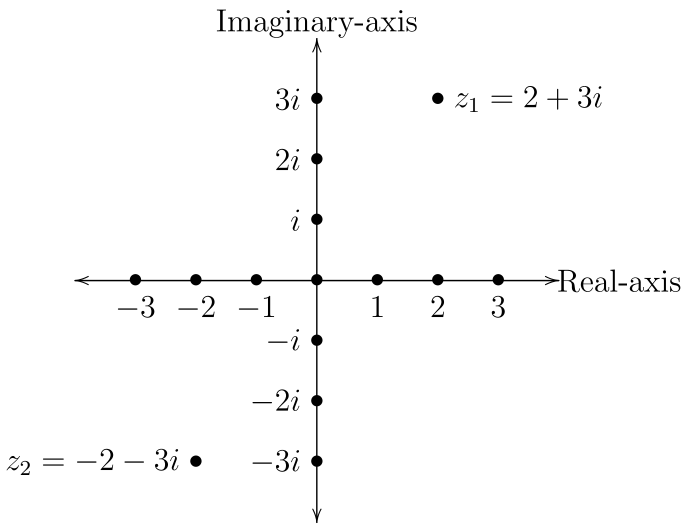
Example 6.6 *Let* $z_1 = 3 + 2i$ *and* $z_2 = 2 + i$. *Then*

$$\frac{z_1}{z_2} = \frac{3+2i}{2+i} = \frac{3+2i}{2+i} \cdot \frac{2-i}{2-i} = \frac{6-3i+4i-2i^2}{4-i^2} = \frac{6+i+2}{5} = \frac{8+i}{5} = \frac{8}{5} + \frac{i}{5}.$$

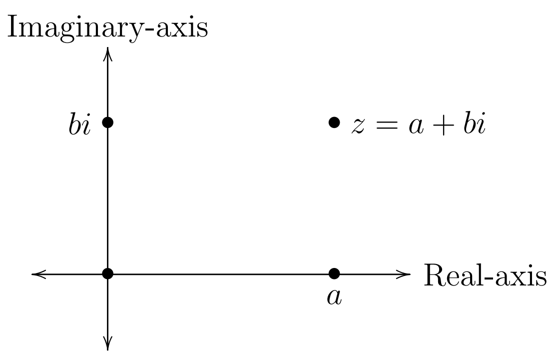
Definition 6.7 *The modulus of a complex number denoted by* $|z|$ *is the distance from the origin to the complex number in an Argand plane.*

We can think of $a + bi$ on an Argand plane corresponding to the point (a, b) on an x, y plane. The difference is that instead of an x-axis, we have a real axis and instead of a y-axis we have an imaginary axis.

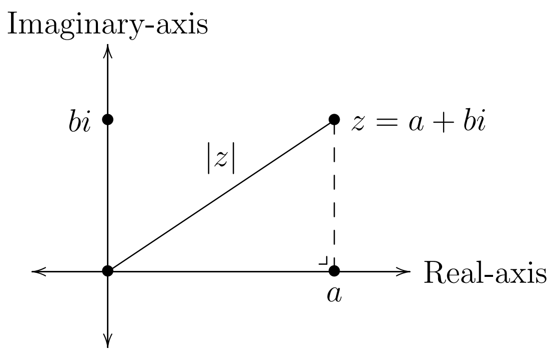
Example 6.8 *Plot* $z_1 = 2 + 3i$ *and* $z_2 = -2 - 3i$ *on an Argand plane.*



Now consider $z = a + bi \in \mathbb{C}$.



If we draw a line from the origin to z (this is $|z|$) and a line parallel to the imaginary axis through z we form a right angled triangle.



If we apply Pythagoras' Theorem

$$|z|^2 = a^2 + b^2$$
$$|z| = \sqrt{a^2 + b^2}.$$

Example 6.9 *Let* $z_1 = 2 + 3i \in \mathbb{C}$ *and* $z_2 = -4 - 3i \in \mathbb{C}$. *Then*

(*i*) $|z_1| = \sqrt{(2)^2 + (3)^2} = \sqrt{4+9} = \sqrt{13}$.

(*i*) $|z_2| = \sqrt{(4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$.

6.1.1 Properties of $\overline{z}$ and $|z|$

In this section we will highlight some important properties of $\overline{z}$ and $|z|$.

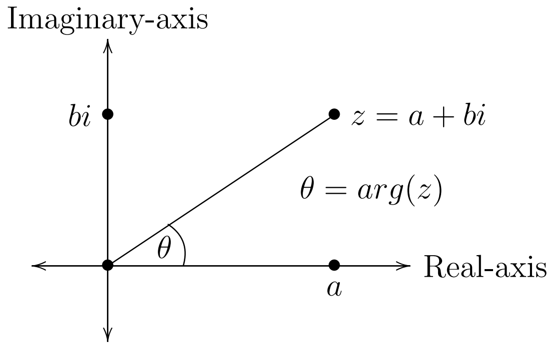
Theorem 6.10 *Let* $z_1, z_2 \in \mathbb{C}$, *then*

1. $z\overline{z} = |z|^2$.
2. $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$.
3. $\overline{z_1.z_2} = \overline{z_1}.\overline{z_2}$.
4. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$.
5. $|z_1.z_2| = |z_1|.|z_2|$.
6. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$.

6.1.2 Argument of a Complex Number

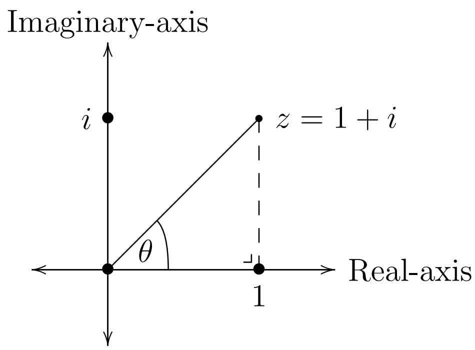
Consider $z = a + bi \in \mathbb{C}$.

Definition 6.11 *If we draw a line from the complex number to the origin. The angle that this line make with the positive real axis is called the argument of* z *(denoted by* $arg(z)$*).*



Example 6.12 *Let $z = 1 + i \in \mathbb{C}$. Find $arg(z)$.*

First we plot z and draw a line parallel to the imaginary axis through z and let $\theta = arg(z)$.



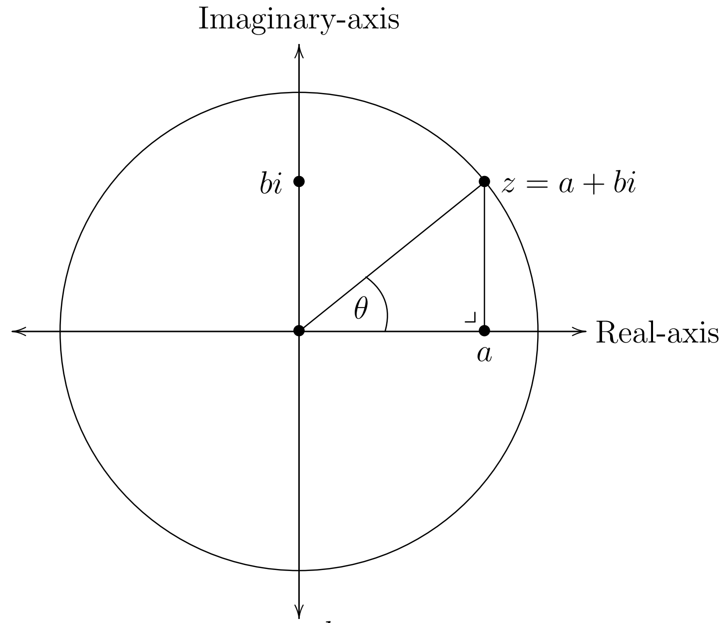
Clearly we have a right angled triangle where the opposite is 1 and the adjacent is 1. Therefore

$$\begin{aligned}\tan(\theta) &= \frac{1}{1} \\ \tan(\theta) &= 1 \\ \theta &= \tan^{-1}(1) \\ \theta &= \frac{\pi}{4} = arg(z).\end{aligned}$$

6.2 Converting Cartesian form to Polar/Exponential form

6.2.1 Converting Cartesian form to Polar form

Consider $z = a + bi$ (this is called cartesian form). If we plot z on an Argand plane, it is equivalent to plotting (a, b) on an x, y plane. Now if instead of thinking of z as a point on an Argand plane, we can think of z being a point on a circle centered at $0 = 0 + 0i$. The obvious question is what attributes do we need to describe z (on a circle). We would certainly need the radius of the circle i.e. the modulus of $|z|$. Also if we draw a line from z to the origin, we need the angle it makes with the positive real axis (i.e. the argument of z). Now our aim is to get a and b in terms of the radius (modulus) and the angle (argument). Let $r = |z|$ and $\theta = arg(z)$.



Then $\cos(\theta) = \frac{a}{r} \iff a = r\cos(\theta)$ and $\sin(\theta) = \frac{b}{r} \iff b = r\sin(\theta)$. There we can rewrite $z = a+bi$ as

$$\begin{aligned} z &= a + bi \\ &= r\cos(\theta) + r\sin(\theta)i \\ &= r(\cos(\theta) + i\sin(\theta)) \end{aligned}$$

where $\theta = arg(z)$ and $r = |z|$. Thus $z = r(\cos(\theta) + i\sin(\theta))$ is called polar form. Now let's take some examples of complex numbers in cartesian form and convert them to polar form.

Example 6.13 *Convert $z = 1 + i \in \mathbb{C}$ to polar form. We know already that $\theta = arg(z) = \frac{\pi}{4}$. Also $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. There when we convert $z = 1 + i$ into cartesian form we get:*

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$$

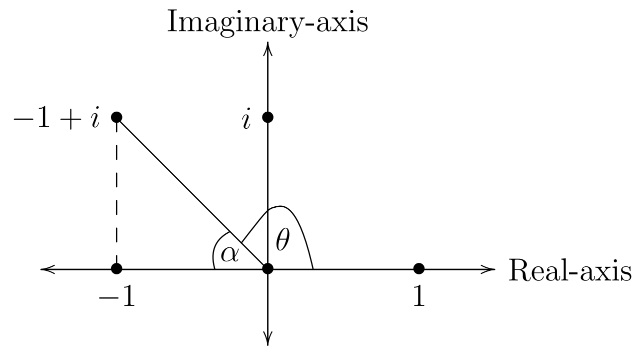
Example 6.14 *Convert $z = -1 + i \in \mathbb{C}$ into polar form.*

Note that we can't calculate $\theta = arg(z)$ directly since $z = -1 + i$ is in the second quadrant. Now if we draw a line from z to the origin and measure the angle that this line makes with the negative real axis (call this α), then we can easily calculate θ by the formula $\theta = \pi - \alpha$ since $\theta + \alpha = \pi$.

(1) $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

(2) $\tan\alpha = \dfrac{\sqrt{1}}{1} = 1$

$\alpha = \tan^{-1}(1) = \dfrac{\pi}{4}$



(3) $\therefore \theta = \pi - \alpha$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

Thus when we convert $-1+i$ to polar form, we get

$$z = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right).$$

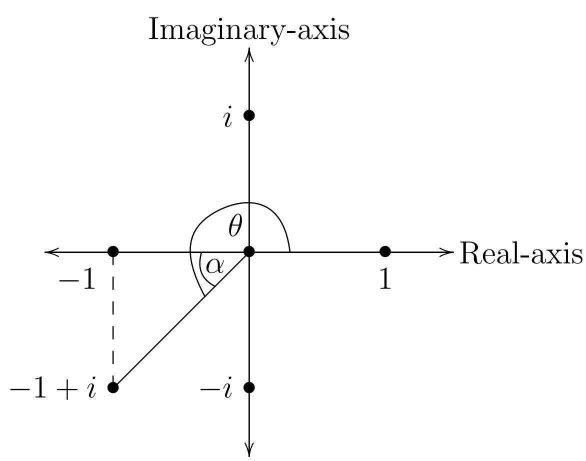
Example 6.15 *Convert $z = 1 - i \in \mathbb{C}$ into polar form.*

Note that we can't calculate $\theta = arg(z)$ directly since $z = 1 - i$ is in the third quadrant.

(1) $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

(2) $\tan\alpha = \dfrac{\sqrt{1}}{1} = 1$

$\alpha = \tan^{-1}(1) = \dfrac{\pi}{4}$



(3) $\therefore \theta = \pi + \alpha$

$$= \pi + \frac{\pi}{4}$$

$$= \frac{5\pi}{4}$$

Thus when we convert $-1-i$ to polar form, we get

$$z = \sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right).$$

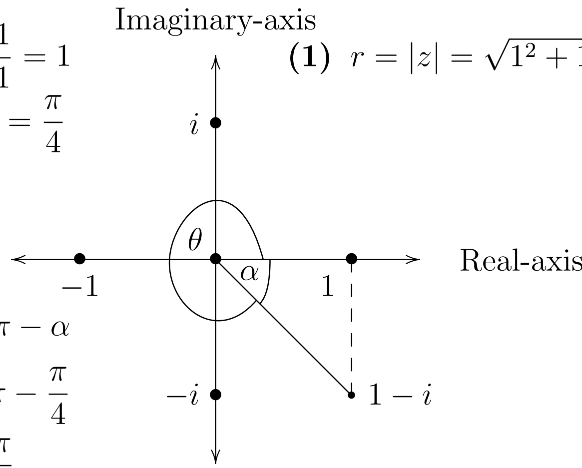
Example 6.16 *Convert $z = -1 - i \in \mathbb{C}$ into polar form.*

Note that we can't calculate $\theta = arg(z)$ directly since $z = -1 - i$ is in the third quadrant.

(1) $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

(2) $\tan\alpha = \frac{1}{1} = 1$

$\alpha = \tan^{-1}(1) = \frac{\pi}{4}$



(3) $\therefore \theta = 2\pi - \alpha$

$= 2\pi - \frac{\pi}{4}$

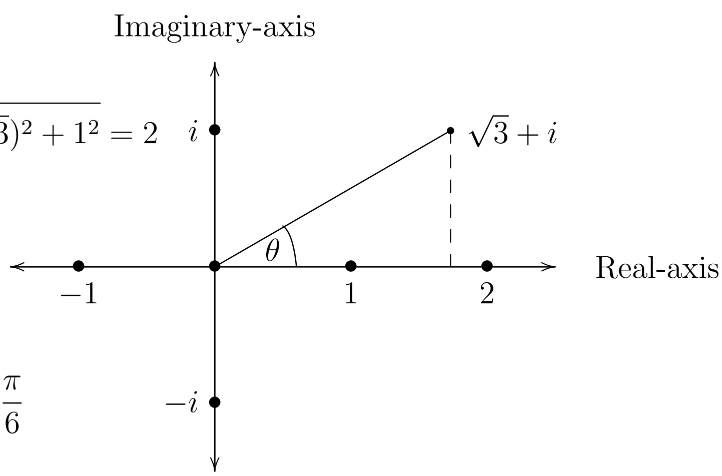
$= \frac{7\pi}{4}$

Thus when we convert $1 - i$ to polar form, we get

$$\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\,\sin\left(\frac{7\pi}{4}\right)\right).$$

Example 6.17 *Convert* $z = \sqrt{3} + i \in \mathbb{C}$ *into polar form.*

(1) $r = |z| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$



(2) $\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Thus when we convert $\sqrt{3} + i$ to polar form, we get

$$2\left(\cos\left(\frac{\pi}{6}\right) + i\,\sin\left(\frac{\pi}{6}\right)\right).$$

Example 6.18 *Convert* $z = -1 + \sqrt{3}i \in \mathbb{C}$ *into polar form.*

(1) $r = |z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

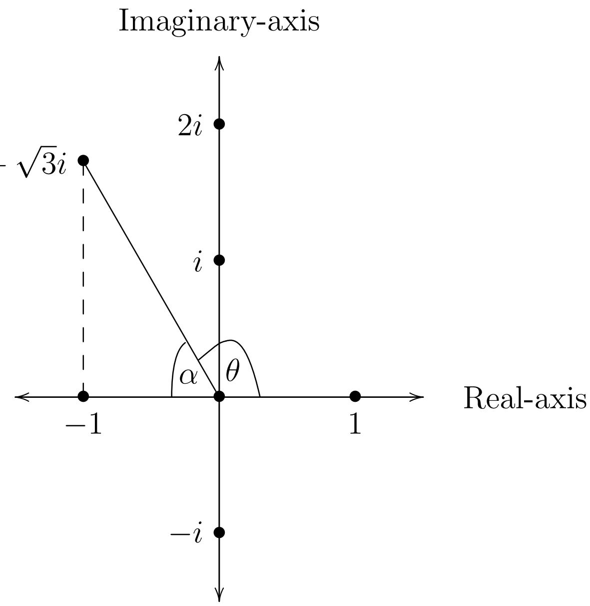
(2) $\tan\alpha = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

$\alpha = \tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$

(3) $\therefore \theta = \pi - \alpha$

$= \pi - \dfrac{\pi}{3}$

$= \dfrac{2\pi}{3}$



Thus when we convert $-1 + \sqrt{3}i$ to polar form, we get

$$2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right).$$

Example 6.19 *Convert* $z = -1 - \sqrt{3}i \in \mathbb{C}$ *into polar form.*

(1) $r = |z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

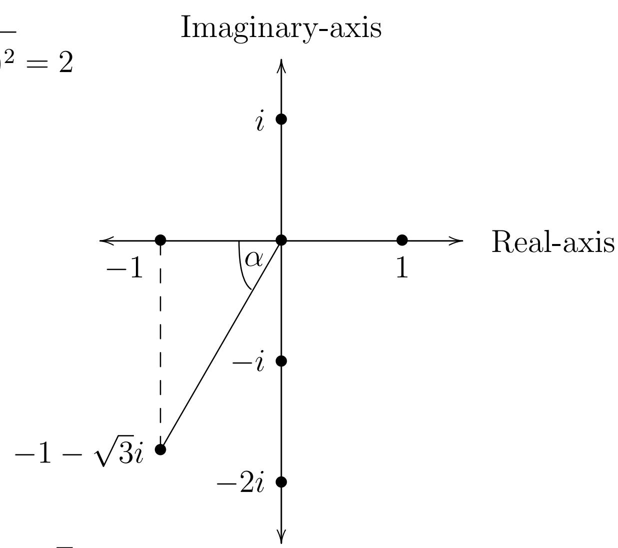
(2) $\tan\alpha = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

$\alpha = \tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$

(3) $\therefore \theta = \pi + \alpha$

$= \pi + \dfrac{\pi}{3}$

$= \dfrac{4\pi}{3}$



Thus when we convert $-1 - \sqrt{3}i$ to polar form, we get

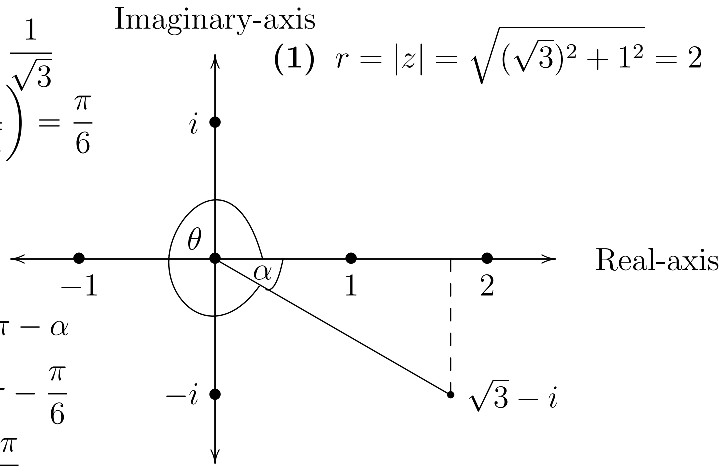
$$2\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right).$$

Example 6.20 *Convert* $z = \sqrt{3} - i \in \mathbb{C}$ *into polar form.*

(1) $r = |z| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

(2) $\tan\alpha = \dfrac{1}{\sqrt{3}}$

$\alpha = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$



(3) $\therefore \theta = 2\pi - \alpha$

$= 2\pi - \dfrac{\pi}{6}$

$= \dfrac{11\pi}{6}$

Thus when we convert $\sqrt{3} - i$ to polar form, we get

$$2\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right).$$

6.2.2 Converting Cartesian form to Exponential form

Consider $z = a + ib \in \mathbb{C}$ (cartesian form). Clearly $z = r(\cos(\theta) + i\sin(\theta))$ (polar form) where $r = |z|$ and $\theta = arg(z)$. There is a well known formula (Euler's formula) that states that:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Therefore we can rewrite $z = r(\cos(\theta) + i\sin(\theta))$ as $z = re^{i\theta}$ where $r = |z|$ and $\theta = arg(z)$. We shall call this exponential form.

Example 6.21 *Convert $z = 1 + i \in \mathbb{C}$ into exponential form.*

We know (from example 13.10) that $r = |z| = \sqrt{2}$ and $\theta = \dfrac{\pi}{4}$. Thus when we convert $1 + i$ into exponential form, we get

$$z = \sqrt{2}e^{i\left(\frac{\pi}{4}\right)} = \sqrt{2}e^{\frac{\pi i}{4}}.$$

Example 6.22 *Convert $z = -1 - \sqrt{3}i \in \mathbb{C}$ into exponential form.*

We know (from example 13.16) that $r = |z| = 2$ and $\theta = \dfrac{4\pi}{3}$. Thus when we convert $-1 - \sqrt{3}i$ into exponential form, we get

$$z = 2e^{i\left(\frac{4\pi}{3}\right)} = 2e^{\frac{4\pi i}{3}}.$$

Please note that it is very useful to be able to be able to go from polar form to exponential form and vice versa.

6.3 Multiplying/Dividing Complex Numbers in Polar Form

Here's some important facts about multiplying/dividing complex numbers in polar form.

Theorem 6.23 *Let* $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)) \in \mathbb{C}$ *and* $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2)) \in \mathbb{C}$. *Then*

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \in \mathbb{C}.$$

At this point let's prove the Theorem. I am going to prove this Theorem both ways to highlight the importance of being able to go from polar form to exponential form and vice versa.

Proof.[Method 1] let $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)) \in \mathbb{C}$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2)) \in \mathbb{C}$. Then

$$\begin{aligned} z_1 z_2 &= r_1(\cos(\theta_1) + i\sin(\theta_1))\, r_2(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2(\cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) + i^2 sin(\theta_1)\sin(\theta_2)) \\ &= r_1 r_2([\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)] + i[\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)]) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

since $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ and $\sin(A+B) = \cos(A)\sin(B) - \cos(A)\sin(B)$. ■

Proof.[Method 2] let $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)) \in \mathbb{C}$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2)) \in \mathbb{C}$. Then

$$\begin{aligned} z_1 z_2 &= r_1(\cos(\theta_1) + i\sin(\theta_1))\, r_2(\cos(\theta_2) + i\sin(\theta_2)) = r_1 e^{i\theta_1}\, r_2 e^{i\theta_2} = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i\theta_1 + i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)} = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)). \end{aligned}$$

■

Clearly it's easier in this case to be able to go from polar form to exponential form and vice versa.

Theorem 6.24 *Let* $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)) \in \mathbb{C}$ *and* $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2)) \in \mathbb{C}$. *Then*

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)) \in \mathbb{C}.$$

Proof.

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos(\theta_1) + i\sin(\theta_1))}{r_2(\cos(\theta_2) + i\sin(\theta_2))} \\ &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} \\ &= \frac{r_1}{r_2}\left(\frac{e^{i\theta_1}}{e^{i\theta_2}}\right) \\ &= \frac{r_1}{r_2} e^{i\theta_1 - i\theta_2} \\ &= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \\ &= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)). \end{aligned}$$

■

Example 6.25 *Let* $z_1 = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$ *and* $z_2 = 3\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$. *Find*

$$(i)\ z_1z_2 \quad (ii)\ \frac{z_1}{z_2}.$$

(*i*)

$$\begin{aligned}
z_1z_2 &= 6\left(\cos\left(\frac{5\pi}{6}+\frac{\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}+\frac{\pi}{6}\right)\right)\\
&= 6\left(\cos\left(\frac{6\pi}{6}\right) + i\sin\left(\frac{6\pi}{6}\right)\right)\\
&= 6\left(\cos\left(\pi\right) + i\sin\left(\pi\right)\right)\\
&= 6(-1+0i)\\
&= -6.
\end{aligned}$$

(*ii*) $\frac{z_1}{z_2}$.

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{2}{3}\left(\cos\left(\frac{5\pi}{6}-\frac{\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}-\frac{\pi}{6}\right)\right)\\
&= \frac{2}{3}\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)\\
&= \frac{2}{3}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\\
&= -\frac{1}{3} - i\frac{\sqrt{3}}{3}\\
&= -\frac{1}{3}(1+i\sqrt{3}).
\end{aligned}$$

6.4 De Moivre's Theorem

Consider $(1+i)^{50}$. Clearly it would be very difficult to work this out by hand. However if we convert it to polar form, we can easily work it out using de Moivre's Theorem.

Theorem 6.26 (de Moivre's Theorem) *If* $z = r(\cos(\theta) + i\sin(\theta)) \in \mathbb{C}$, *then*

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Example 6.27 *Calculate* $(1+i)^{50}$.

Let $z = 1 + i$, when we convert this to polar form we get $z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\cos\left(\frac{\pi}{4}\right)\right)$. Now we shall use de Moivre's Theorem to calculate $(1+i)^{50}$.

$$\begin{aligned} z^{50} &= (\sqrt{2})^{50}\left(\cos\left(\frac{50\pi}{4}\right) + i\sin\left(\frac{50\pi}{4}\right)\right) \\ &= ((\sqrt{2})^2)^{25}\left(\cos\left(\frac{25\pi}{2}\right) + i\sin\left(\frac{25\pi}{2}\right)\right) \\ &= 2^{25}\left(\cos\left(\frac{25\pi}{2}\right) + i\sin\left(\frac{25\pi}{2}\right)\right) \end{aligned}$$

Clearly $\frac{25\pi}{2}$ is angle greater than $360° = 2\pi$. So have to basically peel of layers of 2π to get back down to and angle between 0 and 2π. Now $\frac{4\pi}{2} = 2\pi$. Now $25 = 6 \cdot 4 + 1$ i.e.

$$\frac{25\pi}{2} = 6\left(\frac{4\pi}{2}\right) + \frac{\pi}{2} = 6\,(2\pi) + \frac{\pi}{2} = 6.0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

Therefore

$$\begin{aligned} z^{50} &= 2^{25}\left(\cos\left(\frac{\pi}{2}\right) + i\cos\left(\frac{\pi}{2}\right)\right) \\ &= 2^{25}\,(0 + i) \\ &= 2^{25} i. \end{aligned}$$

Example 6.28 *Calculate* $(-1 + \sqrt{3}i)^{11}$,

When we convert $z = -1 - \sqrt{3}i$ to polar form, we get $z = 2\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$.

$$\begin{aligned} z^{11} &= 2^{11}\left(\cos\left(\frac{44\pi}{3}\right) + i\sin\left(\frac{44\pi}{3}\right)\right) \\ &= 2^{11}\left(\cos\left(7\left(\frac{6\pi}{3}\right) + \frac{2\pi}{3}\right) + i\sin\left(7\left(\frac{6\pi}{3}\right) + \frac{2\pi}{3}\right)\right) \\ &= 2^{11}\left(\cos\left(7\,(2\pi) + \frac{2\pi}{3}\right) + i\sin\left(7\,(2\pi) + \frac{2\pi}{3}\right)\right) \end{aligned}$$

$$\begin{aligned} &= 2^{11}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) \\ &= 2^{11}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= -2^{10} + 2^{10}\sqrt{3}\,i. \end{aligned}$$

Example 6.29 *Calculate* $(-1 + \sqrt{3}i)^{41}$.

When we convert $z = -1 + \sqrt{3}i$ to polar form, we get $z = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$.

$$\begin{aligned}
z^{41} &= 2^{41}\left(\cos\left(\frac{82\pi}{3}\right) + i\sin\left(\frac{82\pi}{3}\right)\right) \\
&= 2^{41}\left(\cos\left(13\left(\frac{6\pi}{3}\right) + \frac{4\pi}{3}\right) + i\sin\left(13\left(\frac{6\pi}{3}\right) + \frac{4\pi}{3}\right)\right) \\
&= 2^{41}\left(\cos\left(13\,(2\pi) + \frac{4\pi}{3}\right) + i\sin\left(13\,(2\pi) + \frac{4\pi}{3}\right)\right) \\
&= 2^{41}\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) \\
&= 2^{41}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \\
&= -2^{40} - 2^{40}\sqrt{3}\,i.
\end{aligned}$$

Exercises 4

Q1 Let $z_1 = \sqrt{3} + i$ and $z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Evaluate the following:

(i) $z_1 + z_2$.
(ii) $z_1 - z_2$.

(iii) $z_1 z_2$.

(iv) $\overline{z_1}$.

(v) $\overline{z_2}$.

(vi) $\frac{z_1}{z_2}$.

(vii) $\frac{z_2}{z_1}$.

(viii) $|z_1|$.

(ix) $|z_2|$.

(x) $\arg(z_1)$.

(xi) $\arg(z_2)$.

(xii) Polar form of z_1 and z_2.

(xiii) Exponential form of z_1 and z_2.

(xiv) ${z_1}^{100}$, using De Moivre's theorem.

(xv) ${z_2}^{50}$, using De Moivre's theorem.

Q2 Let $z_1 = -2 - 2i$ and $z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Evaluate the above.

6.5 Answers

Exercises 4

Q1 (i) $z_1 + z_2 = (\sqrt{3} - \frac{1}{2}) + (\frac{\sqrt{3}}{2} + 1)\, i.$

(ii) $z_1 - z_2 = (\sqrt{3} + \frac{1}{2}) + (1 - \frac{\sqrt{3}}{2})\, i.$

(iii) $z_1 z_2 = -\sqrt{3} + i.$

(iv) $\overline{z_1} = \sqrt{3} - i.$

(v) $\overline{z_2} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i.$

(vi) $\frac{z_1}{z_2} = -2i.$

(vii) $\frac{z_2}{z_1} = \frac{1}{2} i.$

(viii) $|z_1| = 2.$

(ix) $|z_2| = 1.$

(x) $\arg(z_1) = \frac{\pi}{6}.$

(xi) $\arg(z_2) = \frac{4\pi}{6}.$

(xii) $z_1 = 2(\cos\frac{\pi}{6} + i\,\sin\frac{\pi}{6})$ and $z_2 = \cos\frac{4\pi}{6} + i\,\sin\frac{4\pi}{6}.$

(xiii) $z_1 = 2e^{i\frac{\pi}{6}}$ and $z_2 = e^{i\frac{4\pi}{6}}.$

(xiv) ${z_1}^{100} = 2^{100}(-\frac{1}{2} + \frac{\sqrt{3}}{2} i).$

(xv) ${z_2}^{50} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i.$

Q2 (i) $z_1 + z_2 = -1 - (2 + \sqrt{3})\, i.$

(ii) $z_1 - z_2 = -3 + (\sqrt{3} - 2)\, i.$

(iii) $z_1 z_2 = -(2 + 2\sqrt{3}) + (2\sqrt{3} - 2)i.$

(iv) $\overline{z_1} = -2 + 2i.$

(v) $\overline{z_2} = 1 + \sqrt{3}i.$

(vi) $\frac{z_1}{z_2} = \frac{-1 + \sqrt{3}}{2} - \frac{1 + \sqrt{3}}{2}\, i.$

(vii) $\frac{z_2}{z_1} = \frac{-1 + \sqrt{3}}{4} + \frac{1 + \sqrt{3}}{4}\, i.$

(viii) $|z_1| = 2\sqrt{2}.$

(ix) $|z_2| = 2$.

(x) $\arg(z_1) = \dfrac{5\pi}{4}$.

(xi) $\arg(z_2) = \dfrac{5\pi}{3}$.

(xii) $z_1 = 2\sqrt{2}(\cos\dfrac{5\pi}{4} + i\,\sin\dfrac{5\pi}{4})$ and $z_2 = 2(\cos\dfrac{5\pi}{3} + i\,\sin\dfrac{5\pi}{3})$.

(xiii) $z_1 = 2\sqrt{2}e^{i\frac{5\pi}{4}}$ and $z_2 = 2e^{i\frac{5\pi}{3}}$.

(xiv) ${z_1}^{100} = -(2\sqrt{2})^{100}$.

(xv) ${z_2}^{50} = 2^{50}(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i)$.