Maths Notes for first year Engineers

5. Differential Equations

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Chapter 5

Differential Equations

Definition 5.1 *A differential equation is an equation that describes a relationship between a function and the derivatives of that function. The order of the highest derivative in the differential equation defines the order of the differential equation.*

Example 5.2 $\dfrac{dy}{dx} = y$ *is an example of a first order differential equation.*

Example 5.3 $\dfrac{d^2y}{dx^2} = \dfrac{y}{x^2}$ *is an example of a second order differential equation.*

In this chapter we will discuss standard methods for solving first and second order differential equations. We will deal with three types of first order differential equations:

1. First Order-Separable
2. First Order-Homogeneous
3. First Order-Linear.

Also we will look at two types of second order differential equations:

1. Second Order-Homogeneous
2. Second Order-Non Homogeneous.

5.1 First Order-Separable

A first order separable equation takes the form $\dfrac{dy}{dx} = \dfrac{l(x)}{h(y)}$ where l is a function of x and h is a function of y. When solving equations of this form, we manipulate the equation such that all the x-terms are on one side of the equation and the y-terms are on the other. Note that in doing this we must ensure that dy and dx are both above the line. At this point, we integrate both sides and solve for y. i.e.

$$\frac{dy}{dx} = \frac{l(x)}{h(y)} \Longrightarrow h(y)\,dy = l(x)\,dx \Longrightarrow \int h(y)\,dy = \int l(x)\,dx.$$

Example 5.4 *Solve the first order separable differential equation* $\dfrac{dy}{dx} = 6y^2x$ *subject to* $y(0) = 1$.

Solution.

$$\begin{aligned}
\frac{dy}{dx} &= 6y^2x \\
\int \frac{dy}{y^2} &= \int 6x\,dx \\
\int y^{-2}\,dy &= \int 6x\,dx \\
\left[\frac{y^{-1}}{-1}\right] &= 3x^2 + c \\
-\frac{1}{y} &= 3x^2 + c \\
y &= -\left(\frac{1}{3x^2 + c}\right) \\
y(x) &= -\left(\frac{1}{3x^2 + c}\right)
\end{aligned}$$

When we apply our initial condition ($y(0) = 1$), this will enable us to find c.

$$y(0) = -\left(\frac{1}{3(0)^2 + c}\right) \iff 1 = -\left(\frac{1}{c}\right) \iff c = -1.$$

Therefore

$$y(x) = -\left(\frac{1}{3x^2 - 1}\right) = \frac{1}{1 - 3x^2}.$$

♣

Example 5.5 *Solve* $\dfrac{dy}{dx} = \dfrac{x^2y - 4y}{x + 2}$ *given that* $y(0) = 100$.

Solution.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x^2y-4y}{x+2} \\
\frac{dy}{dx} &= \frac{y(x^2-4)}{x+2} \\
\frac{dy}{dx} &= \frac{y(x-2)(x+2)}{x+2} \\
\frac{dy}{dx} &= y(x-2) \\
\int \frac{dy}{y} &= \int (x-2)\,dx \\
\ln|y| &= \frac{x^2}{2} - 2x + c \\
y &= e^{\frac{x^2}{2}-2x+c} \\
y &= e^{\frac{x^2}{2}-2x}e^c \\
y(x) &= A\,e^{\frac{x^2}{2}-2x} \qquad \text{where } A = e^c.
\end{aligned}$$

Now $y(0) = 100$, thus

$$y(0) = A\,e^{\frac{(0)^2}{2}-2(0)} \Longleftrightarrow 100 = Ae^0 \Longleftrightarrow A = 100.$$

Therefore

$$y(x) = 100\,e^{\frac{x^2}{2}-2x}.$$

♣

Example 5.6 *Solve* $x\dfrac{dy}{dx} = 3y + 10$*, given that* $y(1) = -\dfrac{1}{3}$.

Solution.

$$\begin{aligned}
x\frac{dy}{dx} &= 3y + 10 \\
\int \frac{dy}{3y+10} &= \int \frac{dx}{x} \\
\frac{\ln|3y+10|}{3} &= \ln|x| + c \\
3y + 10 &= e^{3\ln|x|+3c} \\
3y &= e^{\ln|x^3|}e^{3c} - 10 \\
3y &= Ax^3 - 10 \\
y(x) &= \frac{Ax^3}{3} - \frac{10}{3}
\end{aligned}$$

If $y(1) = -\dfrac{1}{3}$, then $-\dfrac{1}{3} = \dfrac{A}{3} - \dfrac{10}{3} \Longrightarrow \dfrac{9}{3} = \dfrac{A}{3} \Longrightarrow A = 9$, therefore $y(x) = 3x^3 - \dfrac{10}{3}$. ♣

Exercises 3.1

Q1 Solve the following first order separable differential equations:

(i) $\dfrac{dy}{dx} = \dfrac{x}{y}$.
(ii) $\dfrac{dy}{dx} = \dfrac{y^2}{x+2}$.
(iii) $\dfrac{dy}{dx} = y^{-3}\sin 2x$.

5.2 First Order-Homogeneous

Definition 5.7 *A* ***homogeneous*** *function is a function where the sum of the powers of every term are equal.*

Example 5.8 $f(x, y) = x + y$, $g(x, y) = x^2 + xy + 11y^2$ *and* $h(x, y) = x^3 + x^2y + xy^2 + 7y^3$ *are examples of homogeneous equations of degree* 1, 2 *and* 3 *respectively.*

Example 5.9 $f(x, y) = x^2 + y$, $g(x, y) = x^2 + x + 8y^2$ *are not examples of homogeneous equations.*

A homogeneous first order differential equation is an equation that takes the form:

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree.

When solving differential equations of this type we do the following:

1. Let $y = vx$.
2. By the product rule, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.
3. Replace y with vx and $\dfrac{dy}{dx}$ with $v + x\dfrac{dv}{dx}$ in the original equation. After this substitution, this will transform the Homogeneous into a separable differential equation.
4. Solve this differential equation for v (in terms of x).
5. Let $v = \dfrac{y}{x}$ to obtain y in terms of x.

Example 5.10 *Solve* $\dfrac{dy}{dx} = \dfrac{x+y}{x}$*, given that* $y(1) = 2$.

Solution. Let $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x+y}{x} \\
v + x\frac{dv}{dx} &= \frac{x+vx}{x} \quad [3] \\
v + x\frac{dv}{dx} &= 1 + v \\
\int dv &= \int \frac{dx}{x} \\
v &= \ln|x| + c \quad [4] \\
\frac{y}{x} &= \ln|x| + c. \quad [5] \\
y(x) &= x(\ln|x| + c)
\end{aligned}$$

If $y(1) = 2$, then $2 = 1(\ln(1) + c) \Longrightarrow c = 2$, therefore $y(x) = x(\ln|x| + 2)$. ♣

Example 5.11 *Solve* $\frac{dy}{dx} = \frac{xy - y^2}{x^2}$, *given that* $y(1) = \frac{1}{2}$.

Solution. Let $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{xy - y^2}{x^2} \\
v + x\frac{dv}{dx} &= \frac{x^2 v - x^2 v^2}{x^2} \\
v + x\frac{dv}{dx} &= \frac{x^2(v - v^2)}{x^2} \\
v + x\frac{dv}{dx} &= v - v^2 \\
x\frac{dv}{dx} &= -v^2 \\
-\int \frac{dv}{v^2} &= \int \frac{dx}{x} \\
-\int v^{-2}\, dv &= \int \frac{dx}{x} \\
-\left[\frac{v^{-1}}{-1}\right] &= \ln|x| + c \\
\frac{1}{v} &= \ln|x| + c \\
v &= \frac{1}{\ln|x| + c} \\
\frac{y}{x} &= \frac{1}{\ln|x| + c} \\
y &= \frac{x}{\ln|x| + c}
\end{aligned}
$$

If $y(1) = \frac{1}{2}$, then $\dfrac{1}{2} = \dfrac{1}{\ln(1) + c} \implies c = 2$, therefore $y(x) = \dfrac{x}{\ln|x| + 2}$. ♣

Example 5.12 *Solve* $\dfrac{dy}{dx} = \dfrac{y^2 - xy + x^2}{xy - x^2}$, *given that* $y(1) = 3$.

Solution. Let $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y^2 - xy + x^2}{xy - x^2} \\
v + x\frac{dv}{dx} &= \frac{x^2v^2 - x^2v + x^2}{x^2v - x^2} \\
v + x\frac{dv}{dx} &= \frac{v^2 - v + 1}{v - 1} \\
x\frac{dv}{dx} &= \frac{v^2 - v + 1}{v - 1} - v \\
x\frac{dv}{dx} &= \frac{1}{v - 1} \\
\int v - 1\, dv &= \int \frac{dx}{x}
\end{aligned}$$

$$\begin{aligned}
\frac{v^2}{2} - v &= \ln|x| + c \\
v^2 - 2v &= 2\ln|x| + 2c \\
v^2 - 2v - (2\ln|x| + 2c) &= 0.
\end{aligned}$$

Let $a = 1$, $b = -2$ and $c = -(2\ln|x| + 2c)$. Then

$$\begin{aligned}
v &= \frac{-(-2) \pm \sqrt{4 - 4(1)(-(2\ln|x| + 2c))}}{2(1)} \\
&= \frac{2 \pm \sqrt{4 + 4(2\ln|x| + 2c)}}{2} \\
&= \frac{2 \pm \sqrt{4}\sqrt{1 + 2\ln|x| + 2c}}{2} \\
&= \frac{2 \pm 2\sqrt{1 + 2\ln|x| + 2c}}{2} \\
&= 1 \pm \sqrt{1 + 2\ln|x| + 2c}.
\end{aligned}$$

Now $v = \dfrac{y}{x}$, thus,

$$\begin{aligned}
\frac{y}{x} &= 1 \pm \sqrt{1 + 2\ln|x| + 2c} \\
y &= x(1 \pm \sqrt{1 + 2\ln|x| + 2c})
\end{aligned}$$

$y(1) = 3$, therefore

$$\begin{aligned}
3 &= 1(1 \pm \sqrt{1 + 2\ln|1| + 2c}) \\
3 &= 1 \pm \sqrt{1 + 2c} \\
2 &= \pm\sqrt{1 + 2c} \\
4 &= 1 + 2c \qquad \text{square both sides} \\
c &= \frac{3}{2}.
\end{aligned}$$

Therefore $y = x(1 \pm \sqrt{4 + 2\ln|x|})$. ♣

Exercises 3.2

Q1 Solve the following first order homogeneous differential equations:

(i) $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{x^2}$.
(ii) $\dfrac{dy}{dx} = \dfrac{xy^2 + x^2 y}{x^3}$.

5.3 First Order-Linear

A linear first order differential equation takes the form:

$$\frac{dy}{dx} + p(x)y = q(x) \quad \text{or} \quad y' + p(x)y = q(x)$$

where p and q are functions of x.

When solving equations of this type, we do the following:

1. Identify $p(x)$ and $q(x)$.
2. Calculate $\rho(x)$ (the integrating factor) where $\rho(x) = e^{\int p(x)\,dx}$.
3. Calculate $y(x)$ where
$$y(x) = \frac{1}{\rho(x)} \int \rho(x) q(x)\,dx.$$

Example 5.13 *Solve* $y' - \frac{2y}{x} = x^2 e^x$.

Solution.

$$\begin{aligned}
y' - \frac{2y}{x} &= x^2 e^x \\
y' + y\left(\frac{-2}{x}\right) &= x^2 e^x
\end{aligned}$$

Then $p(x) = \left(\frac{-2}{x}\right)$, $q(x) = x^2 e^x$ [1] and

$$\rho(x) = e^{\int \left(\frac{-2}{x}\right) dx} = e^{-2\int \left(\frac{1}{x}\right) dx} = e^{-2\ln|x|} = e^{\ln|x^{-2}|} = x^{-2} = \frac{1}{x^2}. \quad [2]$$

Therefore

$$\begin{aligned} y(x) &= \frac{1}{\rho(x)} \int \rho(x) q(x)\, dx \\ &= \frac{1}{\frac{1}{x^2}} \int \frac{1}{x^2} x^2 e^x \, dx \quad [3] \\ &= x^2 \int e^x \, dx \\ &= x^2 \left[e^x + c\right]. \end{aligned}$$

♣

Example 5.14 *Solve* $x^2 \frac{dy}{dx} + (2x^2 + 3x)y = xe^{-2x}$.

Solution.

$$\begin{aligned} x^2 \frac{dy}{dx} + (2x^2 + 3x)y &= xe^{-2x} \\ \frac{dy}{dx} + \left(2 + \frac{3}{x}\right) y &= \frac{e^{-2x}}{x} \quad \text{divide across by } x^2 \end{aligned}$$

Then $p(x) = \left(2 + \frac{3}{x}\right)$, $q(x) = \frac{e^{-2x}}{x}$ and

$$\rho(x) = e^{\int \left(2 + \frac{3}{x}\right) dx} = e^{(2x + 3\ln x)} = e^{2x} e^{\ln x^3} = e^{2x} x^3.$$

Therefore

$$\begin{aligned} y(x) &= \frac{1}{\rho(x)} \int \rho(x) q(x)\, dx \\ &= \frac{1}{e^{2x} x^3} \int e^{2x} x^3 x e^{-2x} \, dx \\ &= \frac{1}{x^3 e^{2x}} \int x^2 \, dx \\ &= \frac{1}{x^3 e^{2x}} \left[\frac{x^3}{3} + c\right]. \end{aligned}$$

♣

Example 5.15 *Solve* $x \frac{dy}{dx} - 4y = x^5 \cos(5x)$.

Solution.

$$\begin{aligned} x\frac{dy}{dx} - 4y &= x^5\cos(5x) \\ \frac{dy}{dx} - \frac{4y}{x} &= x^4\cos(5x) \qquad \text{divide across by } x \\ \frac{dy}{dx} + y\left(-\frac{4}{x}\right) &= x^4\cos(5x) \end{aligned}$$

Then $p(x) = \left(\frac{-4}{x}\right)$, $q(x) = x^4\cos(5x)$ and

$$\rho(x) = e^{\int\left(\frac{-4}{x}\right)dx} = e^{-4\int\left(\frac{1}{x}\right)dx} = e^{-4\ln|x|} = e^{\ln|x^{-4}|} = x^{-4} = \frac{1}{x^4}.$$

Therefore

$$\begin{aligned} y(x) &= \frac{1}{\rho(x)}\int \rho(x)q(x)\,dx \\ &= \frac{1}{\frac{1}{x^4}}\int \frac{1}{x^4}x^4\cos(5x)\,dx \\ &= x^4\int \cos(5x)\,dx \\ &= x^4\left[\frac{\sin(5x)}{5} + c\right]. \end{aligned}$$

♣

Exercises 3.3

Q1 Solve the following linear first order differential equations:

(i) $\frac{dy}{dx} + 4xy = x$.

(ii) $\frac{dy}{dx} - 2y = 3e^x$.

(iii) $2\frac{dy}{dx} + y = 3x$.

5.4 Second Order-Homogeneous

A homogeneous second order differential equation takes the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad (\diamond)$$

where a, b and c are constants.

With every differential equation of the form $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$, there exists a corresponding auxiliary equation ($am^2 + bm + c = 0$). Clearly there are 3 possibilities for the solutions of a quadratic equation (real, equal or imaginary roots). Depending on the solutions of the auxiliary equation, they will dictate the form of the solution to the homogeneous second order differential equation ($\diamond$).

- If the roots of the Auxiliary equation are real (i.e. say m_1 and m_2), the solution ($\diamond$) is
$$y(x) = Ae^{m_1 x} + Be^{m_2 x}$$
where A and B are constants.

- If the roots of the Auxiliary equation are equal (i.e. say m), the solution ($\diamond$) is
$$y(x) = e^{mx}(A + Bx)$$
where A and B are constants.

- If the roots of the Auxiliary equation are imaginary (i.e. say $m = \alpha \pm \beta i$), the solution ($\diamond$) is
$$y(x) = e^{\alpha x}(A_1 \cos(\beta x) + A_2 \sin(\beta x))$$
where A_1 and A_2 are constants.

Example 5.16 *Solve* $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 0.$

Solution. The Auxiliary equation is: $m^2 - m - 6 = 0$. The solutions to this equations are:
$$m^2 - m - 6 = 0 \Longrightarrow (m-3)(m+2) = 0 \Longrightarrow m = 3,\ m = -2.$$
Therefore
$$y(x) = Ae^{3x} + Be^{-2x}.$$

♣

Example 5.17 *Solve* $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0,$ *given that* $y(0) = 1$ *and* $y'(0) = 0$.

Solution. The Auxiliary equation is: $m^2 - 4m + 4 = 0$. The solutions to this equations are:
$$m^2 - 4m + 4 = 0 \Longrightarrow (m-2)(m-2) = 0 \Longrightarrow m = 2.$$
Therefore
$$y(x) = e^{2x}(A + Bx).$$
Now we shall use the initail conditions provided to calculate A and B. $y(0) = 1 \Longrightarrow 1 = e^0(A + B(0)) \Longrightarrow A = 1$, therefore $y(x) = e^{2x}(1 + Bx)$.

Now $y'(x) = e^{2x}(B) + (A + Bx)2e^{2x}$ and $y'(0) = 0$. Thus $B + (1 + 0)(2) = 0 \Longrightarrow B = -2$. Therefore
$$y(x) = e^{2x}(1 - 2x).$$

♣

Example 5.18 *Solve* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 26y = 0$, *given that* $y(0) = 1$ *and* $y'(0) = 0$.

Solution. The Auxiliary equation is: $m^2 - 2m + 26 = 0$. The solutions to this equations are:

$$\begin{aligned} m &= \frac{2 \pm \sqrt{-100}}{2} \\ &= \frac{2 \pm \sqrt{100}\sqrt{-1}}{2} \\ &= \frac{2 \pm 10i}{2} \\ &= 1 \pm 5i. \end{aligned}$$

Therefore $y(x) = e^x(A_1 \cos(5x) + A_2 \sin(5x)) = A_1 e^x \cos(5x) + A_2 e^x \sin(5x)$.

$y(0) = 1 \Longrightarrow 1 = A_1 e^0 \cos(0) + A_2 e^0 \sin(0) \Longrightarrow A_1 = 1$, therefore $y(x) = e^x \cos(5x) + A_2 e^x \sin(5x)$.

$y'(x) = -5e^x \sin(5x) + e^x \cos(5x) + 5A_2 e^x \cos(5x) + A_2 e^x \sin(5x)$.
$y'(0) = 0 \Longrightarrow -5e^0 \sin(0) + e^0 \cos(0) + 5A_2 e^0 \cos(0) + A_2 e^0 \sin(0) = 0 \Longrightarrow A_2 = -\frac{1}{5}$. Therefore

$$y(x) = e^x \cos(5x) - \frac{1}{5} e^x \sin(5x).$$

♣

Exercises 3.4

Q1 Solve the following second order homogeneous differential equations:

(i) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 0$.
(ii) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.
(iii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$.

5.5 Second Order-Non Homogeneous

A non-homogenous second order differential equation takes the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b and c are constants and f is a function of x. The solution to this equation is $y = y_H(x) + y_P(x)$ where $y_H(x)$ corresponds to the solution to the auxiliary equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ and $y_P(x)$ is the particular solution and $y_P(x)$ depends on $f(x)$. We shall calculate $y_P(x)$ for $f(x) = Ae^{kx}$, $f(x) = P_n(x)$ ($P_n(x)$ is a polynomial of degree n) and $f(x) = A\sin(\gamma x) + B\cos(\gamma x)$.

5.5.1 $f(x) = Ae^{kx}$

Consider the equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = Ae^{kx}$. First we calculate $y_H(x)$ and then $y_P(x)$. We know how to calculate $y_H(x)$. When $f(x) = Ae^{kx}$, then $y_P(x)$ is of the form Qe^{kx}. Note that $y_P(x)$ also satisfies $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = Ae^{kx}$ (i.e. $ay''_P(x) + by'_P(x) + cy_P(x) = Ae^{kx}$). We use this fact to calculate Q and hence $y_P(x)$.

Example 5.19 *Solve* $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 7e^x$.

Solution. The Auxiliary equation is: $m^2 - 4m + 4 = 0 \iff (m-2)^2 = 0$. The solution to this equation is: $m = 2$.Thus $y_H = e^{2x}(A + Bx)$.

Let $y_P = Qe^x$. Then $y'_P = Qe^x$ and $y''_P = Qe^x$. y_P has to satisfy the above differential equation, therefore

$$\begin{aligned} y''_P - 4y'_P + 4y_P &= 7e^x \\ Qe^x - 4(Qe^x) + 4(Qe^x) &= 7e^x \\ Qe^x &= Qe^x \\ Q &= 7 \quad \text{divide across by } e^x \end{aligned}$$

Thus $y_P = 7e^x$. Therefore $y = y_H + y_P = e^{2x}(A + Bx) + 7e^x$. ♣

Example 5.20 *Solve* $25\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + y = 10e^{7x}$.

Solution. The Auxiliary equation is: $25m^2 - 10m + 1 = 0 \iff (5m-1)^2 = 0$. The solution to this equation is: $m = \frac{1}{5}$. Thus $y_H = e^{\frac{x}{5}}(A + Bx)$.

Let $y_P = ke^{7x}$. Then $y'_P = 7ke^{7x}$ and $y''_P = 49ke^{7x}$. y_P has to satisfy the above differential equation, therefore

$$\begin{aligned} 25y''_P - 10y'_P + y_P &= 10e^{7x} \\ 25(49ke^{7x}) - 10(7ke^{7x}) + (ke^7x) &= 10e^{7x} \\ 1225ke^{7x} - 70ke^{7x} + kke^{7x} &= 10e^{7x} \\ 1156ke^{7x} &= 10e^{7x} \\ 1156k &= 10 \quad \text{didvide across by } e^x \\ k &= \frac{5}{578}. \end{aligned}$$

Thus $y_P = \dfrac{5e^{7x}}{578}$. Therefore $y = y_H + y_P = e^{\frac{x}{5}}(A+Bx) + \dfrac{5e^{7x}}{578}$. ♣

5.5.2 $f(x) = P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

When $f(x) = P_n(x)$ ($P_n(x)$ is a polynomial of degree n) then we let $y_P(x) = T_n(x)$ (i.e. we let $y_P(x)$ be a general polynomial of the same degree of $P_n(x)$. Again $y_P(x)$ also satisfies the original differential equation and we use this fact to find the unknown coefficients. If $f(x) = 3 + 4x$ (polynomial of degree 1), then we would let $y_P(x) = a + bx$ and find a and b using the fact that satisfies the original differential equation. If $f(x) = 3 - x^2$ then we would let $y_P(x) = a + bx + cx^2$ and find a, b and c using the fact that satisfies the original differential equation.

Example 5.21 *Solve* $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 7y = 5x.$

Solution. The Auxiliary equation is: $m^2 - 8m + 7 = 0 \Longleftrightarrow (m-1)(m-7) = 0$. The solutions to this equation are: $m_1 = 1$ and $m_2 = 7$. Thus $y_H = Ae^{7x} + Be^x$.

Let $y_P = a_0 + a_1x$. Then $y'_P = a_1$ and $y''_P = 0$. y_P has to satisfy the above differential equation, therefore

$$\begin{aligned} y''_P - 8y'_P + 7y_P &= 5x \\ 0 - 8(a_1) + 7(a_0 + a_1x) &= 5x \\ -8a_1 + 7a_0 + 7a_1x &= 5x \\ (-8a_1 + 7a_0) + 7a_1x &= 0 + 5x \end{aligned}$$

Therefore $-8a_1 + 7a_0 = 0$ and $7a_1 = 5$. Therefore $a_1 = \dfrac{5}{7}$ and $a_0 = \dfrac{40}{49}$. Thus $y_P = \dfrac{40}{49} + \dfrac{5x}{7}$. Therefore $y = y_H + y_P = Ae^{7x} + Be^x + \dfrac{40}{49} + \dfrac{5x}{7}$. ♣

Example 5.22 *Solve* $2\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 3y = x^2 + 1.$

Solution. The Auxiliary equation is: $2m^2 - 5m - 3 = 0 \Longleftrightarrow (2m-3)(m-1) = 0$. The solutions to this equation are: $m_1 = \dfrac{3}{2}$ and $m_2 = 1$. Thus $y_H = Ae^{\frac{3x}{2}} + Be^x$.

Let $y_P = a_0 + a_1x + a_2x^2$. Then $y'_P = a_1 + 2a_2x$ and $y''_P = 2a_2$. y_P has to satisfy the above differential equation, therefore

$$\begin{aligned} 2y''_P - 5y'_P + 3y_P &= x^2 + 1 \\ 2(2a_2) - 5(a_1 + 2a_2x) + 3(a_0 + a_1x + a_2x^2) &= x^2 + 1 \\ 4a_2 - 5a_1 - 10a_2x + 3a_0 + 3a_1x + 3a_2x^2 &= x^2 + 1 \\ x^2[3a_2] + x[-10a_2 + 3a_1] + [4a_2 - 5a_1 + 3a_0] &= x^2 + 0x + 1 \end{aligned}$$

Therefore $3a_2 = 1 \Longrightarrow a_2 = \dfrac{1}{3}$, $-10a_2 + 3a_1 = 0$ and $4a_2 - 5a_1 + 3a_0 = 1$.

$$\begin{aligned}
-10a_2 + 3a_1 &= 0 \\
3a_1 &= 10a_2 \\
a_1 &= \frac{10a_2}{3} \\
a_1 &= \frac{10}{9}.
\end{aligned}$$

$$\begin{aligned}
4a_2 - 5a_1 + 3a_0 &= 1 \\
\frac{4}{3} - \frac{50}{9} + 3a_0 &= 1 \\
-\frac{38}{9} + 3a_0 &= 1 \\
3a_0 &= \frac{47}{9} \\
a_0 &= \frac{47}{27}.
\end{aligned}$$

Thus $y_P = \frac{47}{27} + \frac{10x}{9} + \frac{x^2}{3}$. Therefore $y = y_H + y_P = Ae^{\frac{3x}{2}} + Be^x + \frac{47}{27} + \frac{10x}{9} + \frac{x^2}{3}$. ♣

5.5.3 $f(x) = A\sin(\gamma x) + B\cos(\gamma x)$

When $f(x) = A\sin(\gamma x) + B\cos(\gamma x)$ (i.e. $f(x) = A\sin(\gamma x)$ or $f(x) = B\cos(\gamma x)$ or $f(x) = A\sin(\gamma x) + B\cos(\gamma x)$), we let $y_P(x) = A_1\sin(\gamma x) + A_2\cos(\gamma x)$. Again $y_P(x)$ also satisfies the original differential equation and we use this fact to find A_1 and A_2.

Example 5.23 *Solve* $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = \cos(x)$.

Solution. The Auxiliary equation is: $4m^2 - 12m + 9 = 0 \iff (2m-3)^2 = 0$. The solution to this equation is: $m = \frac{3}{2}$. Thus $y_H = e^{\frac{3x}{2}}(A + Bx)$.

Let $y_P = A_1\cos(x) + A_2\sin(x)$. Then $y'_P = -A_1\sin(x) + A_2\cos(x)$ and $y''_P = -A_1\cos(x) - A_2\sin(x)$. y_P has to satisfy the above differential equation, therefore

$$\begin{aligned}
4y''_P - 12y'_P + 9y &= \cos(x) \\
4(-A_1\cos(x) - A_2\sin(x)) - 12(-A_1\sin(x) + A_2\cos(x)) + 9(A_1\cos(x) + A_2\sin(x)) &= \cos(x) \\
-4A_1\cos(x) - 4A_2\sin(x) + 12A_1\sin(x) - 12A_2\cos(x) + 9A_1\cos(x) + 9A_2\sin(x) &= \cos(x) \\
\cos(x)(-4A_1 - 12A_2 + 9A_1) + \sin(x)(-4A_2 + 12A_1 + 9A_2) &= \cos(x) \\
\cos(x)(5A_1 - 12A_2) + \sin(x)(12A_1 + 5A_2) &= \cos(x) + 0\sin(x)
\end{aligned}$$

Thus $5A_1 - 12A_2 = 1$ and $12A_1 + 5A_2 = 0$. If we solve these equations, we get that $A_1 = \frac{5}{169}$ and $A_2 = -\frac{12}{169}$. Thus $y_P = \frac{5}{169}\cos(x) - \frac{12}{169}\sin(x)$.

Therefore $y = y_H + y_P = e^{\frac{3x}{2}}(A + Bx) + \frac{5}{169}\cos(x) - \frac{12}{169}\sin(x)$. ♣

Example 5.24 *Solve* $5\frac{d^2y}{dx^2} - 16\frac{dy}{dx} + 3y = \cos(x) + \sin(x)$.

Solution. The Auxiliary equation is: $5m^2 - 16m + 3 = 0 \Longleftrightarrow (5m-1)(m-3) = 0$. The solutions to this equation are: $m_1 = \frac{1}{5}$ and $m_2 = 3$. Thus $y_H = Ae^{3x} + Be^{\frac{x}{5}}$.

Let $y_P = A_1\cos(x) + A_2\sin(x)$. Then $y'_P = -A_1\sin(x) + A_2\cos(x)$ and $y''_P = -A_1\cos(x) - A_2\sin(x)$. Y_E has to satisfy the above differential equation, therefore

$$\begin{aligned} 5y''_P - 16y'_P + 3y &= \cos(x) + \sin(x) \\ \cos(x)(-2A_1 - 16A_2) + \sin(x)(16A_1 - 2A_2) &= \cos(x) + \sin(x) \end{aligned}$$

Thus $-2A_1 - 16A_2 = 1$ and $16A_1 - 2A_2 = 1$. If we solve these equations, we get that $A_1 = \frac{7}{130}$ and $A_2 = -\frac{9}{130}$. Thus $y_P = \frac{7}{130}\cos(x) - \frac{9}{130}\sin(x)$.

Therefore $y = y_H + y_P = Ae^{3x} + Be^{\frac{x}{5}} + \frac{7}{130}\cos(x) - \frac{9}{130}\sin(x)$. ♣

Exercises 3.5

Q1 Solve the following second order homogeneous differential equations:

(i) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 3e^{-2x}$.

(ii) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x^2 + 4x + 1$.

(iii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10\cos x$.

5.6 Answers

Exercises 3.1

Q1 (i) $y^2(x) = \dfrac{x^2}{2} + 2c$, (ii) $y(x) = \dfrac{1}{c - \ln|x+2|}$, (iii) $y^4(x) = -2\cos 2x + c$.

Exercises 3.2

Q1 (i) $y(x) = x\tan(\ln|x| + c)$, (ii) $y(x) = \dfrac{x}{\ln|x| + c}$.

Exercises 3.3

Q1 (i) $y(x) = ce^{-2x^2} + \dfrac{1}{4}$, (ii) $y(x) = ce^{2x} - 3ye^{x}$, (iii) $y(x) = ce^{-\frac{x}{2}} + 3x - 6$.

Exercises 3.4

Q1 (i) $y(x) = c_1e^x + c_2e^{-4x}$, (ii) $y(x) = e^{2x}(c_1 + xc_2)$, (iii) $y(x) = e^x(c_1\cos 2x + c_2\sin 2x)$.

Exercises 3.5

Q1 (i) $y(x) = c_1e^x + c_2e^{-4x} - \dfrac{1}{2}e^{-2x}$, (ii) $y(x) = e^{2x}(c_1 + xc_2) + \dfrac{1}{2}x^2 + 2x + 2$,
(iii) $y(x) = e^x(c_1\cos 2x + c_2\sin 2x) + 2\cos x - \sin x$.