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Maths Notes for first year Engineers

4. Integration

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Chapter 4

Integration

4.1 Indefinite and Definite Integrals

4.1.1 Indefinite Integrals

When asked to integrate $f(x)$, we write $\displaystyle\int f(x)\,dx$. So $\displaystyle\int \diamond\,dx$ represents applying the process of integration to $\diamond$. Here are some basic rules of integration:

$f(x)$	$\int f(x)\,dx$
x^n	$\frac{x^{n+1}}{n+1} \quad n \neq -1$
$\frac{1}{x}$	$\ln x$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
e^x	e^x

Example 4.1 *Evaluate* $\displaystyle\int x^5\,dx$.

$\displaystyle\int x^5\,dx = \frac{x^6}{6} + c$ where c is a constant of integration.

Example 4.2 *Evaluate* $\displaystyle\int \sqrt{x}\,dx$.

$$\int \sqrt{x}\, dx = \int x^{\frac{1}{2}}\, dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}x^{\frac{3}{2}} + c.$$

Example 4.3 *Evaluate* $\int \frac{1}{x^2}\, dx.$

$$\int \frac{1}{x^2}\, dx = \int x^{-2}\, dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c.$$

Example 4.4 *Evaluate* $\int 1\, dx.$

$$\int 1\, dx = \int x^0\, dx = \frac{x^1}{1} + c = x + c.$$

It is important to note that $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$ i.e. the integral of a sum of functions is equal to the sum of the integrals of the functions. Also $\int k\, f(x) dx = k \int f(x)\, dx$ i.e. the integral of a constant times a function is equal to the constant times the integral of the function.

Example 4.5 *Evaluate* $\int \left(7x^5 + \frac{7}{x} + \frac{8}{x^3} + 10\sin x + 3e^x\right)\, dx.$

$$\int \left(7x^5 + \frac{7}{x} + \frac{8}{x^3} + 10\sin x + 3e^x\right)\, dx = \int \left(7\left(x^5\right) + 7\left(\frac{1}{x}\right) + 8(x^{-3}) + 10(\sin x) + 3(e^x)\right)\, dx =$$
$$7\left(\frac{x^6}{6}\right) + 7\ln x + 8\left(\frac{x^{-2}}{-2}\right) - 10\cos x + 3e^x + c = \frac{7}{6}x^6 + 7\ln x - \frac{4}{x^2} - 10\cos x + 3e^x + c.$$

Example 4.6 *Evaluate* $\int \left(\frac{3}{x^5} + 7 + \frac{3}{\sqrt{x}} + 3\sin x\right)\, dx.$

$$\int \left(\frac{3}{x^5} + 7 + \frac{3}{\sqrt{x}} + 3\sin x\right)\, dx = \int \left(3(x^{-5}) + 7(1) + 3\left(x^{-\frac{1}{2}}\right) + 3(\sin x)\right)\, dx$$
$$= 3\left(\frac{x^{-4}}{-4}\right) + 7x + 3\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right) - 3\cos x + c = -\frac{3}{4x^4} + 7x + 6x^{\frac{1}{2}} - 3\cos x + c.$$

Example 4.7 *Evaluate* $\int \left(\frac{11}{x} + 8 + 7\cos x + 11e^x\right)\, dx.$

$$\int \left(\frac{11}{x} + 8 + 7\cos x + 11e^x\right)\, dx = \int \left(11\left(\frac{1}{x}\right) + 8(1) + 7(\cos x) + 11(e^x)\right)\, dx$$
$$= 11\ln x + 8x + 7\sin x + 11e^x + c.$$

Example 4.8 *Evaluate* $\int \left(\frac{4}{x^5} + \frac{10}{x} + 11 + 3\sin x + 3\cos x + 4e^x \right) dx.$

$$\int \left(\frac{4}{x^5} + \frac{10}{x} + 11 + 3\sin x + 3\cos x + 4e^x \right) dx = \int \left(4x^{-5} + 10\left(\frac{1}{x}\right) + 11 + 3\sin x + 3\cos x + 4e^x \right) dx$$
$$= \frac{4x^{-4}}{-4} + 10\ln x + 11x - 3\cos x + 3\sin x + 4e^x + c = \frac{1}{x^4} + 10\ln x + 11x - 3\cos x + 3\sin x + 4e^x + c.$$

Exercises 2.1.1

Q1 Evaluate the following:

(i) $\int \left(x^3 + 7\sin x + 8\cos x\right) dx.$

(ii) $\int \left(\frac{7}{x^4} - \frac{7}{x^5} - \frac{1}{\sqrt{x}} + 7 \right) dx.$

(iii) $\int \left(\frac{1}{x^7} + \frac{11}{x} + 13\cos x + 7e^x \right) dx.$

(iv) $\int \left(21\cos x - \frac{2}{x} + x^{11} - \frac{4}{x^4} + 21 \right) dx.$

4.1.2 Definite Integrals

We may be asked to calculate the integral of a function between two values a and b. This type of integral is called definite integral and is denoted as $\int_a^b f(x)\,dx$. The difference between a definite integral and an indefinite integral is that the former is a number whereas the latter is a function of x. Let's calculate the following definite integrals.

Example 4.9 *Evaluate* $\int_1^2 \left(x^2 + x + 1\right) dx.$

$\int_1^2 \left(x^2 + x + 1\right) dx = \left[\frac{x^3}{3} + \frac{x^2}{2} + x + c \right]_1^2$. Now we calculate [sub in 2 for x] $-$ [sub in 1 for x].

$$\left[\frac{x^3}{3} + \frac{x^2}{2} + x + c \right]_1^2 = \left[\frac{2^3}{3} + \frac{2^2}{2} + 2 + c \right] - \left[\frac{1^3}{3} + \frac{1^2}{2} + 1 + c \right] = \frac{8}{3} + 4 + c - \frac{1}{3} - \frac{1}{2} - 1 - c = \frac{29}{40}.$$

When dealing with definite integrals, clearly c will always disappear/cancel. So from this point on we will not include it when evaluating definite integrals.

Example 4.10 *Evaluate* $\int_0^1 \left(\frac{1}{\sqrt{x}} + 7 + e^x \right) dx.$

$$\int_0^1 \left(\frac{1}{\sqrt{x}} + 7 + e^x\right) dx = \int_0^1 \left(x^{-\frac{1}{2}} + 7 + e^x\right) dx = \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + 7x + e^x\right]_0^1 = \left[2x^{\frac{1}{2}} + 7x + e^x\right]_0^1$$
$$= (2\sqrt{1} + 7 + e) - (0 + 0 + 1) = 8 + e.$$

Example 4.11 *Evaluate* $\int_1^3 \left(\frac{3}{x^4} - \frac{3}{x} + \frac{e^x}{3}\right) dx$.

$$\int_1^3 \left(\frac{3}{x^4} - \frac{3}{x} + \frac{e^x}{3}\right) dx = \int_1^3 \left(3x^{-4} - 3\left(\frac{1}{x}\right) + \frac{1}{3}\left(e^x\right)\right) dx = \left[\left[\frac{3x^{-3}}{-3}\right] - 3\ln x + \frac{1}{3}\left(e^x\right)\right]_1^3$$
$$= \left[-\frac{1}{x^3} - 3\ln x + \frac{e^x}{3}\right]_1^3 = \left[-\frac{1}{27} - 3\ln 3 + \frac{e^3}{3}\right] - \left[-\frac{1}{1} - 3\ln 1 + \frac{e}{3}\right] = \frac{26}{27} - 3\ln 3 + \left(\frac{e^3 - e}{3}\right).$$

Example 4.12 *Evaluate* $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)\ dx$.

$$\int_0^{\frac{\pi}{2}} (\sin x + \cos x)\ dx = \Big[-\cos x + \sin x\Big]_0^{\frac{\pi}{2}} = \left[-\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] - [-\cos 0 + \sin 0] = [0 + 1] - [-1 + 0] = 2.$$

Exercises 2.1.2

Q1 Evaluate the following:

(i) $\int_1^2 \left(x^3 + x^2 + x\right)\ dx$.

(ii) $\int_0^1 \left(\frac{1}{\sqrt{x}} + 11 + 3e^x\right)\ dx$.

(iii) $\int_1^2 \left(\frac{2}{x^3} - \frac{4}{x} + 3\right)\ dx$.

(iv) $\int_0^{\pi} (\sin x + \cos x)\ dx$.

4.2 Integration by parts

The integration by parts formula is the following.

$$\int f'(x)g(x)\, dx = f(x)g(x) - \int f(x)g'(x)\, dx.$$

Our main aim here is to write one of the two function as the derivative of a function. Let's see an example.

Example 4.13 *Evaluate* $\int x\cos x\, dx$.

In this example the integrated quantity is the product of two basic functions, which we know how to integrate. It is convenient to write $\cos x$ as the derivative of another function. We know that the intagral of $\cos x$ is $\sin x$. Therefore, let $g(x) = x$ and $f(x) = \sin x$, so that

$$\int x \cos x \, dx = \int x (\sin x)' \, dx.$$

Now we can apply the integration by parts formula to evaluate the integral.

$$\int x(\sin x)' \, dx = x \sin x - \int (x)' \sin x \, dx = x \sin x - \int 1 \cdot \sin x \, dx = x \sin x - \int \sin x \, dx$$
$$= x \sin x - (-\cos x) + c = x \sin x + \cos x + c.$$

Example 4.14 *Evaluate* $\int 3xe^{2x} \, dx$.

It is convenient here to write e^{2x} as the derivative of another function. We know that the integral of e^x is e^x. Therefore, let $g(x) = x$ and $f'(x) = e^{2x}$, so that $f(x) = \frac{1}{2}e^{2x}$ and thus

$$\int 3xe^{2x} \, dx = \int 3x(\frac{1}{2}e^{2x})' \, dx.$$

Now we can apply the integration by parts formula to evaluate the integral.

$$\int 3x(\frac{1}{2}e^{2x})' \, dx = 3x\frac{1}{2}e^{2x} - \int (3x)'\frac{1}{2}e^{2x} \, dx = \frac{3}{2}xe^{2x} - \int \frac{3}{2}e^{2x} \, dx$$
$$\frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x} + c.$$

Exercises 2.2

Q1 Calculate the following:

(i) $\int x \sin 4x \, dx$.

(ii) $\int 2x \ln x \, dx$.

(iii) $\int xe^x \, dx$.

(iv) $\int e^x \cos x \, dx$.

(v) $\int x^2 e^x \, dx$.

(vi) $\int xe^{-x} \, dx$.

4.3 Integration By Substitution I

As a way to introduce integration by substitution, we will solve the following:

- $\int (ax+b)^n \, dx \quad n \neq 1.$
- $\int \frac{1}{ax+b} \, dx.$
- $\int \sin(ax+b) \, dx.$
- $\int \cos(ax+b) \, dx.$
- $\int e^{ax+b} \, dx.$

At this moment, we cannot use any rules in the tables to calculate any of the above. The idea behind integration by substitution is that after we make a certain substitution, we can then use one of rules discussed in the previous section to solve the question.

(1) $\int (ax+b)^n \, dx, \quad n \neq 1$. Let $u = ax+b \Longrightarrow \frac{du}{dx} = a \Longrightarrow dx = \frac{du}{a}$. After we make the substitution, we get

$$\int u^n \frac{du}{a} = \frac{1}{a} \int u^n du = \frac{1}{a}\left[\frac{u^{n+1}}{n+1}\right] + c = \frac{u^{n+1}}{a(n+1)} + c.$$

Now replace u with $ax+b$, therefore $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c.$

Example 4.15 *Evaluate* $\int (3x-7)^5 \, dx$.

$$\int (3x-7)^5 \, dx = \frac{(3x-7)^6}{3 \cdot 6} + c = \frac{(3x-7)^6}{18} + c.$$

Example 4.16 *Evaluate* $\int \frac{1}{(2x+3)^3} \, dx$.

$$\int \frac{1}{(2x+3)^3} \, dx = \int (2x+3)^{-3} \, dx = \frac{(2x+3)^{-2}}{2(-2)} + c = -\frac{1}{4(2x+3)^2} + c.$$

Example 4.17 *Evaluate* $\int_0^1 \frac{1}{\sqrt{2x+2}} \, dx$.

$$\int_0^1 \frac{1}{\sqrt{2x+2}}\,dx = \int_0^1 (2x+2)^{-\frac{1}{2}}\,dx = \left[\frac{(2x+2)^{\frac{1}{2}}}{2\left(\frac{1}{2}\right)}\right]_0^1 = \left[\sqrt{2x+2}\,\right]_0^1 = \sqrt{4}-\sqrt{2} = 2-\sqrt{2}.$$

(2) $\displaystyle\int \frac{1}{ax+b}\,dx$. Let $u = ax+b \Longrightarrow \dfrac{du}{dx} = a \Longrightarrow dx = \dfrac{du}{a}$. After we make the substitution, we get

$$\int \frac{1}{u}\frac{du}{a} = \frac{1}{a}\int \frac{1}{u}du = \frac{1}{a}\left[\ln|u|\right] + c = \frac{\ln|u|}{a} + c.$$

Now replace u with $ax+b$, therefore $\displaystyle\int \frac{1}{ax+b}\,dx = \frac{\ln|ax+b|}{a} + c.$

Example 4.18 *Evaluate* $\displaystyle\int \frac{1}{3x+3}\,dx.$

$$\int \frac{1}{3x+3}\,dx = \frac{\ln|3x+3|}{3} + c.$$

Example 4.19 *Evaluate* $\displaystyle\int_0^1 \frac{1}{2x+1}\,dx.$

$$\int_0^1 \frac{1}{2x+1}\,dx = \left[\frac{\ln|2x+1|}{2}\right]_0^1 = \frac{\ln|2|}{2} - \frac{\ln|1|}{2} = \frac{\ln|2|}{2}.$$

(3) $\displaystyle\int \sin(ax+b)\,dx$. Let $u = ax+b \Longrightarrow \dfrac{du}{dx} = a \Longrightarrow dx = \dfrac{du}{a}$. After we make the substitution, we get

$$\int \sin(u)\frac{du}{a} = \frac{1}{a}\int \sin(u)du = \frac{1}{a}\left[-\cos(u)\right] + c = \frac{-\cos(u)}{a} + c.$$

Now replace u with $ax+b$, therefore $\displaystyle\int \sin(ax+b)\,dx = \frac{-\cos(ax+b)}{a} + c.$

Example 4.20 *Evaluate* $\displaystyle\int \sin(5x-7)\,dx.$

$$\int \sin(5x-7)\,dx = \frac{-\cos(5x-7)}{5} + c.$$

Example 4.21 *Evaluate* $\displaystyle\int_0^{\frac{\pi}{2}} \sin(2x)\,dx.$

$$\int_0^{\frac{\pi}{2}} \sin(2x)\,dx = \left[\frac{-\cos(2x)}{2}\right]_0^{\frac{\pi}{2}} = \left[\frac{-\cos(\pi)}{2} - \left(\frac{-\cos(0)}{2}\right)\right] = \frac{1}{2} + \frac{1}{2} = 1.$$

(4) $\displaystyle\int \cos(ax+b)\,dx$. Let $u = ax+b \Longrightarrow \dfrac{du}{dx} = a \Longrightarrow dx = \dfrac{du}{a}$. After we make the substitution, we get

$$\int \cos(u)\frac{du}{a} = \frac{1}{a}\int \cos(u)du = \frac{1}{a}\left[\sin(u)\right] + c = \frac{\sin(u)}{a} + c.$$

Now replace u with $ax+b$, therefore $\displaystyle\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + c.$

Example 4.22 *Evaluate* $\displaystyle\int \cos(7x-5)\,dx.$

$$\int \cos(7x-5)\,dx = \frac{\sin(7x-5)}{7} + c.$$

Example 4.23 *Evaluate* $\displaystyle\int_0^{\frac{\pi}{4}} \cos(2x)\,dx.$

$$\int_0^{\frac{\pi}{4}} \cos(2x)\,dx = \left[\frac{\sin(2x)}{2}\right]_0^{\frac{\pi}{4}} = \left[\frac{\sin\left(\frac{\pi}{2}\right)}{2} - \frac{\sin(0)}{2}\right] = \frac{1}{2} - 0 = \frac{1}{2}.$$

(5) $\displaystyle\int e^{ax+b}\,dx$. Let $u = ax+b \Longrightarrow \dfrac{du}{dx} = a \Longrightarrow dx = \dfrac{du}{a}$. After we make the substitution, we get

$$\int e^u\frac{du}{a} = \frac{1}{a}\int e^u du = \frac{1}{a}\left[e^u\right] + c = \frac{e^u}{a} + c.$$

Now replace u with $ax+b$, therefore $\displaystyle\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + c.$

Example 4.24 *Evaluate* $\displaystyle\int e^{11x-15}\,dx.$

$$\int e^{11x-15}\,dx = \frac{e^{11x-15}}{11} + c.$$

Example 4.25 *Evaluate* $\displaystyle\int_0^1 e^{x-1}\,dx.$

$$\int_0^1 e^{x-1}\,dx = \left[e^{x-1}\right]_0^1 = e^0 - e^{-1} = 1 - \frac{1}{e}.$$

Exercises 2.3

Q1 Calculate the following:

(i) $\displaystyle\int \frac{1}{\sqrt{x-3}}\,dx.$

(ii) $\displaystyle\int_1^2 (3x-2)^4\,dx.$

(iii) $\displaystyle\int_0^2 \frac{1}{(2x+2)^6}\,dx.$

(iv) $\displaystyle\int_0^1 \frac{1}{2x+2}\,dx$.

(v) $\displaystyle\int \frac{1}{7-4x}\,dx$.

(vi) $\displaystyle\int \sin(2x+5)\,dx$.

(vii) $\displaystyle\int_0^{\frac{\pi}{3}} \sin(3x)\,dx$.

(viii) $\displaystyle\int \cos(3x+11)\,dx$.

(ix) $\displaystyle\int_0^{\frac{\pi}{4}} \sin(4x)\,dx$.

(x) $\displaystyle\int_0^{\frac{1}{4}} e^{4x}\,dx$.

(xi) $\displaystyle\int e^{2x+3}\,dx$.

4.4 Partial Fractions

In this section we solve $\displaystyle\int \frac{ax+b}{(x-\alpha)(x-\beta)}\,dx$. First we use partial fractions to split $\displaystyle\frac{ax+b}{(x-\alpha)(x-\beta)}$ into $\displaystyle\frac{A}{x-\alpha}+\frac{B}{x-\beta}$. Then we integrate these parts individually.

Example 4.26 $\displaystyle\int \frac{1}{x^2-3x+2}\,dx$.

First we factorise the denominator $\displaystyle\frac{1}{x^2-3x+2}=\frac{1}{(x-1)(x-2)}$. Then, we split up the fraction as follows.

$$\frac{1}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2} \quad\Longrightarrow\quad 1=A(x-2)+B(x-1).$$

We need to determine the values of A and B. The above equation must stand for all possible values of x. So we can try convenient values of x.

Let $\underline{x=2}$.

$$1=A(2-2)+B(2-1)$$
$$1=B(1)$$
$$B=1$$

Let $\underline{x=1}$.

$$1=A(1-2)+B(2-2)$$
$$1=A(-1)$$
$$A=-1$$

Therefore $\displaystyle\frac{1}{x^2-3x+2}=-\frac{1}{x-1}+\frac{1}{x-2}$. Thus

$$\int \frac{1}{x^2-3x+2}\,dx=\int\left[-\frac{1}{x-1}+\frac{1}{x-2}\right]dx=-\int\frac{1}{x-1}\,dx+\int\frac{1}{x-2}\,dx=-\ln|x-1|+\ln|x-2|+c.$$

Example 4.27 $\displaystyle\int_0^1 \frac{2x}{x^2+3x+2}\,dx$.

First we factorise the denominator $\dfrac{2x}{x^2+3x+2} = \dfrac{2x}{(x+1)(x+2)}$. Then, we split up the fraction as follows.

$$\frac{2x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \quad \Longrightarrow \quad 2x = A(x+2) + B(x+1).$$

Let $\underline{x=-2}$.

$$\begin{aligned} 2(-2) &= A(-2+2) + B(-2+1) \\ -4 &= B(-1) \\ B &= 4 \end{aligned}$$

Let $\underline{x=-1}$.

$$\begin{aligned} 2(-1) &= A(-1+2) + B(-1+1) \\ -2 &= A(1) \\ A &= -2 \end{aligned}$$

Therefore $\dfrac{2x}{x^2+3x+2} = -2\left(\dfrac{1}{x+1}\right) + 4\left(\dfrac{1}{x+2}\right)$. Thus

$$\int_0^1 \frac{2x}{x^2+3x+2}\,dx = \int_0^1 \left[-2\left(\frac{1}{x+1}\right) + 4\left(\frac{1}{x+2}\right)\right] dx = -2\int_0^1 \frac{1}{x+1}\,dx + 4\int_0^1 \frac{1}{x+2}\,dx$$

$= \big[-2\ln|x+1| + 4\ln|x+2|\big]_0^1 = [-2\ln|2| + 4\ln|3|] - [-2\ln|1| + 4\ln|2|]$
$= -2\ln 2 + 4\ln 3 - 4\ln 2 = 4\ln 3 - 6\ln 2.$

Example 4.28 $\displaystyle\int_2^3 \frac{2x+2}{x^2+2x-3}\,dx$.

First we factorise the denominator $\dfrac{2x+2}{x^2+2x-3} = \dfrac{2x+2}{(x+3)(x-1)}$. Then, we split up the fraction as follows.

$$\frac{2x+2}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1} \quad \Longrightarrow \quad 2x+2 = A(x-1) + B(x+3).$$

Let $\underline{x=1}$.

$$\begin{aligned} 2(1)+2 &= A(1-1) + B(1+3) \\ 4 &= B(4) \\ B &= 1 \end{aligned}$$

Let $\underline{x=-3}$.

$$\begin{aligned} 2(-3)+2 &= A(-3-3) + B(-3+3) \\ -4 &= A(-4) \\ A &= 1 \end{aligned}$$

Therefore $\dfrac{2x+2}{x^2+2x-3} = \dfrac{1}{x+3} + \dfrac{1}{x-1}$. Thus

$$\int_2^3 \frac{2x+2}{x^2+2x-3}\,dx = \int_2^3 \left[\frac{1}{x+3} + \frac{1}{x-1}\right] dx = \int_2^3 \frac{1}{x+3}\,dx + \int_2^3 \frac{1}{x-1}\,dx = \big[\ln|x+3| + \ln|x-1|\big]_2^3$$

$= [\ln 6 + \ln 2] - [\ln|5| + \ln|1|] = \ln 6 + \ln 2 - \ln 5.$

Exercises 2.4

Q1 Calculate the following:
(i) $\displaystyle\int \frac{1}{x^2 - 4x + 3}\,dx$
(ii) $\displaystyle\int_0^1 \frac{2x}{x^2 + 4x + 3}\,dx$
(iii) $\displaystyle\int_2^3 \frac{2x+2}{x^2 + 3x - 4}\,dx.$

4.5 Integration By Substitution II

In this section we discuss integration by substitution in more detail.

Example 4.29 $\displaystyle\int 3x^2(x^3+7)^4\,dx.$

Let $u = x^3 + 7 \Longrightarrow \dfrac{du}{dx} = 3x^2 \Longrightarrow du = 3x^2\,dx$. Therefore

$$\int 3x^2(x^3+7)^4\,dx = \int u^4\,du = \frac{u^5}{5} + c = \frac{(x^3+7)^5}{5} + c.$$

Example 4.30 $\displaystyle\int_0^1 2x\sqrt{x^2+1}\,dx.$

Let $u = x^2 + 1 \Longrightarrow \dfrac{du}{dx} = 2x \Longrightarrow du = 2x\,dx$. Therefore

$$\int_0^1 2x\sqrt{x^2+1}\,dx = \int u^{\frac{1}{2}}\,du = \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] = \left[\frac{2(x^2+1)^{\frac{3}{2}}}{3}\right]_0^1 = \frac{2(2)^{\frac{3}{2}}}{3} - \frac{2}{3} = \frac{2(2^{\frac{3}{2}}-1)}{3}.$$

or

$$\int_0^1 2x\sqrt{x^2+1}\,dx = \int_1^2 u^{\frac{1}{2}}\,du = \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] = \left[\frac{2u^{\frac{3}{2}}}{3}\right]_1^2 = \frac{2(2)^{\frac{3}{2}}}{3} - \frac{2}{3} = \frac{2(2^{\frac{3}{2}}-1)}{3}.$$ (i.e. change the limits also ($u = 1^2 + 1 = 2$, $u = 0^2 + 1 = 1$ and finish the question in the changed variable.)

Example 4.31 $\displaystyle\int_0^1 \frac{x^2}{(x^3+1)^4}\,dx.$

Let $u = x^3 + 1 \Longrightarrow \dfrac{du}{dx} = 3x^2 \Longrightarrow \dfrac{du}{3} = x^2\,dx$. Therefore

$$\int_0^1 \frac{x^2}{(x^3+1)^4}\,dx = \int \frac{1}{u^4}\frac{du}{3} = \frac{1}{3}\int u^{-4}\,du = \frac{1}{3}\left[\frac{u^{-3}}{-3}\right] = \left[-\frac{1}{9(x^3+1)^3}\right]_0^1 = \left(-\frac{1}{72}\right) - \left(-\frac{1}{9}\right) = \frac{7}{72}.$$

or

$$\int_0^1 \frac{x^2}{(x^3+1)^4}\,dx = \int_1^2 \frac{1}{u^4}\frac{du}{3} = \frac{1}{3} = \int_1^2 u^{-4}\,du = \frac{1}{3}\left[\frac{u^{-3}}{-3}\right]_1^2 = \left[-\frac{1}{9u^3}\right]_1^2 = \left(-\frac{1}{72}\right) - \left(-\frac{1}{9}\right) = \frac{7}{72}.$$

Example 4.32 $\displaystyle\int_0^1 \frac{x^2}{x^3+7}\,dx.$

Let $u = x^3+7 \Longrightarrow \dfrac{du}{dx} = 3x^2 \Longrightarrow \dfrac{du}{3} = x^2\,dx$. Therefore

$$\int_0^1 \frac{x^2}{x^3+7}\,dx = \int \frac{1}{u}\frac{du}{3} = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}[\ln u] = \Big[\ln|x^3+7|\Big]_0^1 = \frac{\ln 8 - \ln 7}{3}.$$

or

$$\int_0^1 \frac{x^2}{x^3+7}\,dx = \int_7^8 \frac{1}{u}\frac{du}{3} = \frac{1}{3}\int_7^8 \frac{1}{u}\,du = \frac{1}{3}\Big[\ln u\Big]_7^8 = \frac{\ln 8 - \ln 7}{3}.$$

Example 4.33 $\displaystyle\int_3^4 \frac{4x-6}{x^2-3x+1}\,dx.$

Let $u = x^2-3x+1 \Longrightarrow \dfrac{du}{dx} = 2x-3 \Longrightarrow 2\,du = (4x-6)\,dx$. Therefore

$$\int_3^4 \frac{4x-6}{x^2-3x+1}\,dx = \int \frac{1}{u}(2\,du) = 2\int \frac{1}{u}\,du = 2\Big[\ln u\Big] = 2\Big[\ln(x^2-3x+1)\Big]_3^4 = 2\ln 5.$$

or

$$\int_3^4 \frac{4x-6}{x^2-3x+1}\,dx = \int \frac{1}{u}(2\,du) = 2\int \frac{1}{u}\,du = 2\Big[\ln u\Big]_1^5 = 2\ln 5.$$

Example 4.34 $\displaystyle\int (2x+1)\sin(x^2+x+7)\,dx.$

Let $u = x^2+x+7 \Longrightarrow \dfrac{du}{dx} = 2x+1 \Longrightarrow du = (2x+1)\,dx$. Therefore

$$\int (2x+1)\sin(x^2+x+7)\,dx = \int \sin u\,du = -\cos u + c = -\cos(x^2+x+7) + c.$$

Example 4.35 $\displaystyle\int \left(\frac{1}{x}+e^x\right)\sin(\ln x + e^x)\,dx.$

Let $u = \ln x + e^x \Longrightarrow \dfrac{du}{dx} = \dfrac{1}{x} + e^x \Longrightarrow du = \left(\dfrac{1}{x}+e^x\right)\,dx$. Therefore

$$\int \left(\frac{1}{x}+e^x\right)\sin(\ln x + e^x)\,dx = \int \sin u\,du = -\cos u + c = -\cos(\ln x + e^x) + c.$$

Example 4.36 $\displaystyle\int x^4\cos(x^5+7)\,dx.$

Let $u = x^5+7 \Longrightarrow \dfrac{du}{dx} = 5x^4 \Longrightarrow \dfrac{du}{5} = x^4\,dx$. Therefore

$$\int x^4\cos(x^5+7)\,dx = \int \cos u\,\frac{du}{5} = \frac{1}{5}\int \cos u\,du = \frac{\sin u}{5} + c = \frac{\sin(x^5+7)}{5} + c.$$

Example 4.37 $\displaystyle\int \left(3x^2 + e^x\right)\cos(x^3 + e^x)\,dx.$

Let $u = x^3 + e^x \Longrightarrow \dfrac{du}{dx} = 3x^2 + e^x \Longrightarrow du = \left(3x^2 + e^x\right)\,dx$. Therefore

$$\int \left(3x^2 + e^x\right)\cos(x^3 + e^x)\,dx = \int \cos u\,du = \sin u + c = \sin(x^3 + e^x) + c.$$

Example 4.38 $\displaystyle\int (2x + e^x + 1)e^{x^2+e^x+x}\,dx.$

Let $u = x^2 + e^x + x \Longrightarrow \dfrac{du}{dx} = 2x + e^x + 1 \Longrightarrow du = (2x + e^x + 1)\,dx$. Therefore

$$\int (2x + e^x + 1)e^{x^2+e^x+x}\,dx = \int e^u\,du = e^u + c = e^{x^2+e^x+x} + c.$$

Example 4.39 $\displaystyle\int xe^{3x^2}\,dx.$

Let $u = 3x^2 \Longrightarrow \dfrac{du}{dx} = 6x \Longrightarrow \dfrac{du}{6} = x\,dx$. Therefore

$$\int xe^{3x^2}\,dx = \int e^u\,\frac{du}{6} = \frac{1}{6}\int e^u\,du = \frac{e^u}{6} + c = \frac{e^{3x^2}}{6} + c.$$

Exercises 2.5

Q1 Evaluate the following:

(i) $\displaystyle\int 4x^3(x^4 + 9)^5\,dx.$

(ii) $\displaystyle\int_0^1 3x^2\sqrt{x^3 + 3}\,dx.$

(iii) $\displaystyle\int_0^1 \frac{x}{(x^2 + 1)^3}\,dx.$

(iv) $\displaystyle\int_0^1 \frac{x^3}{x^4 + 7}\,dx.$

(v) $\displaystyle\int_0^1 \frac{6x^2 + 4x}{x^3 + x^2 + 1}\,dx.$

(vi) $\displaystyle\int (3x^2\sin(x^3 + 7)\,dx.$

(vii) $\displaystyle\int (2x + e^x)\sin(x^2 + e^x)\,dx.$

(viii) $\displaystyle\int x^5\cos(x^6 + 7)\,dx.$

(ix) $\displaystyle\int \left(3x^2 + \frac{1}{x}\right)\cos(x^3 + \ln x)\,dx.$

(x) $\displaystyle\int x^3e^{4x^4}\,dx.$

(xi) $\int (1+e^x)e^{x+e^x}\,dx$.

4.6 Trigonometric Integration

We have already integrated some trig functions. However we do need to discuss some tricks needed to integrate some more complicated ones. We need the following trig identities to proceed: $\sin^2 x = \frac{1}{2}(1-\cos(2x))$, $\cos^2 x = \frac{1}{2}(1+\cos(2x))$, $\sin^2 x = 1-\cos^2 x$ and $\cos^2 x = 1-\sin^2 x$.

Example 4.40 $\int \sin^2 x\,dx$.

$$\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos(2x))\,dx = \frac{1}{2}\int (1-\cos(2x))\,dx = \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + c.$$

Example 4.41 $\int_{\frac{\pi}{4}}^{\pi} \cos^2 x\,dx$.

$$\int_{\frac{\pi}{4}}^{\pi} \cos^2 x\,dx = \int_{\frac{\pi}{4}}^{\pi} \frac{1}{2}(1+\cos(2x))\,dx = \frac{1}{2}\int_{\frac{\pi}{4}}^{\pi} (1+\cos(2x))\,dx = \frac{1}{2}\left[x + \frac{\sin(2x)}{2}\right]_{\frac{\pi}{4}}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{\sin\left(\frac{\pi}{2}\right)}{2}\right] - \frac{1}{2}\left[\pi + \frac{\sin(2\pi)}{2}\right] = \frac{\pi}{8} + \frac{1}{4} - \frac{\pi}{2} = \frac{1}{4} - \frac{3\pi}{8}.$$

Example 4.42 $\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$.

$\int_0^{\frac{\pi}{2}} \sin^3 x\,dx = \int_0^{\frac{\pi}{2}} \sin x(\sin^2 x)\,dx = \int_0^{\frac{\pi}{2}} \sin x(1-\cos^2 x)\,dx = \int_0^{\frac{\pi}{2}} \sin x(1-(\cos x)^2)\,dx$. Let $u = \cos x$

$\implies \frac{du}{dx} = -\sin x \implies -du = \sin x\,dx$. Also change the limits. $u = \cos\frac{\pi}{2} = 0$ and $u = \cos 0 = 1$.

Thus $\int_0^{\frac{\pi}{2}} \sin x(1-(\cos x)^2)\,dx = \int_1^0 (1-u^2)(-du) = \int_1^0 (u^2-1)\,du = \left[\frac{u^3}{3} - u\right]_1^0 = 0 - \left[\frac{1}{3} - 1\right] = \frac{2}{3}$.

Example 4.43 $\int \cos^3 x\,dx$.

$\int \cos^3 x\,dx = \int \cos x(\cos^2 x)\,dx = \int \cos x(1-\sin^2 x)\,dx = \int \cos x(1-(\sin x)^2)\,dx$. Let $u = \sin x$

$\implies \frac{du}{dx} = \cos x \implies du = \cos x\,dx$.

Thus $\int \cos x(1-(\sin x)^2)\,dx = \int (1-u^2)\,du = \left[u - \frac{u^3}{3}\right] + c = \sin x - \frac{\sin^3 x}{3} + c$.

Example 4.44 $\displaystyle\int \sin x \cos^3 x \, dx$.

$\displaystyle\int \sin x \cos^3 x \, dx = \int \sin x (\cos x)^3 \, dx$. Let $u = \cos x \implies \dfrac{du}{dx} = -\sin x \implies -du = \sin x \, dx$. Thus $\displaystyle\int \sin x (\cos x)^3 \, dx = \int u^3(-du) = -\frac{u^4}{4} + c = -\frac{\cos^4 x}{4} + c$.

Example 4.45 $\displaystyle\int_0^{\frac{\pi}{2}} \sin^7 x \cos x \, dx$.

$\displaystyle\int_0^{\frac{\pi}{2}} \sin^7 x \cos x \, dx = \int_0^{\frac{\pi}{2}} (\sin x)^7 \cos x \, dx$. Let $u = \sin x \implies \dfrac{du}{dx} = \cos x \implies du = \cos x \, dx$. Also change the limits. $u = \sin \frac{\pi}{2} = 1$ and $u = \sin 0 = 0$. Thus

$$\int_0^{\frac{\pi}{2}} (\sin x)^7 \cos x \, dx = \int_0^1 u^7 \, du = \left[\frac{u^8}{8}\right]_0^1 = \frac{1}{8}.$$

Exercises 2.6

Q1 Evaluate the following trigonometric integrals:
(i) $\displaystyle\int_0^{\pi} \sin^2 x \, dx$.
(ii) $\displaystyle\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \cos^2 x \, dx$.
(iii) $\displaystyle\int_0^{\pi} \sin^3 x \, dx$.
(iv) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$.
(v) $\displaystyle\int_0^{\pi} \sin x \cos^7 x \, dx$.
(vi) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 x \cos x \, dx$.

4.7 Area

Using integration, we can find the area included between a function $f(x)$ and the x-axis from $x = a$ to $x = b$ by the formula $\displaystyle\int_a^b f(x) \, dx$ or $\displaystyle\int_a^b y \, dx$.

If the function $f(x)$ is below the x-axis and we want to find the area between the function $f(x)$ and the x-axis from $x = a$ to $x = b$, we use the formula $\left|\displaystyle\int_a^b f(x) \, dx\right|$ or $\left|\displaystyle\int_a^b y \, dx\right|$.

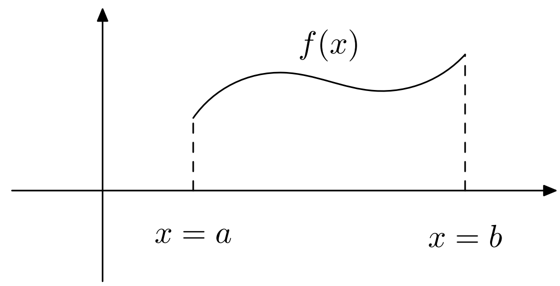

Figure 4.1: Area between $f(x)$ and x-axis from $x = a$ to $x = b$.

When you are calculating areas using integration, it is important to roughly sketch the function before applying any of the above formulae.

Example 4.46 *Find the area between the x-axis and $y = x$ from $x = -1$ to $x = 1$.*

If we sketch $y = x$ from $x = -1$ and $x = 1$ and shade in the area that we are looking for, we get:

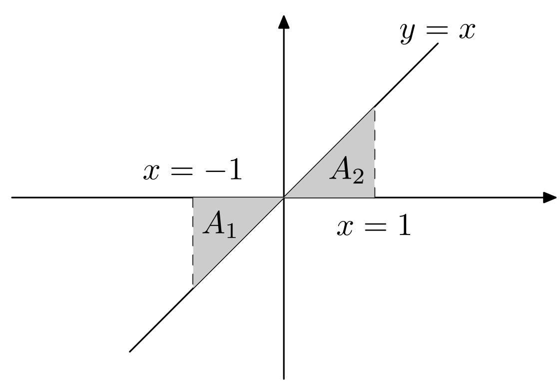

Figure 4.2: Area between $y = x$ and x-axis from $x = 1$ to $x = -1$.

We use the $\left|\int_a^b y\,dx\right|$ to find A_1 and $\int_a^b y\,dx$ to find A_2 since A_1 is below the x-axis and A_2 is above the x-axis. Therefore

$$A_1 + A_2 = \left|\int_{-1}^0 x\,dx\right| + \int_0^1 x\,dx = \left|\left[\frac{x^2}{2}\right]_{-1}^0\right| + \left[\frac{x^2}{2}\right]_0^1 = \left|0 - \frac{1}{2}\right| + \left(\frac{1}{2} - 0\right) = \frac{1}{2} + \frac{1}{2} = 1.$$

Example 4.47 *Find the area enclosed between the $y = x^2$ and $y = x$.*

First we find the points of intersections. Let $x^2 = x \Longrightarrow x^2 - x = 0 \Longrightarrow x(x-1) = 0 \Longrightarrow x = 0$ and $x = 1$. When $x = 0 \Longrightarrow y = 0$ and $x = 1 \Longrightarrow y = 1$. Thus the points of intersection are $(0, 0)$ and $(1, 1)$. Now $y = x$ is a line and $y = x^2$ is U-shaped curve. Here is a sketch of the area between these two curves:
The problem is that we can't directly calculate this area since integration only allows us to find the area between a function and the x-axis between two points. To calculate the shaded region, first we

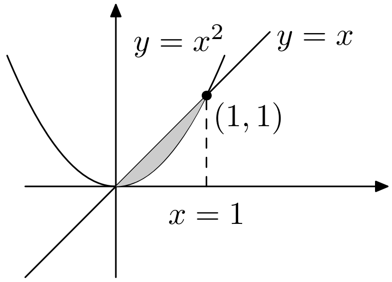

Figure 4.3: Area between $y = x^2$ and $y = x$.

find the area between $y = x$ and the x-axis from $x = 0$ to $x = 1$ (area A_1). We then find the area between $y = x^2$ and the x-axis from $x = 0$ to $x = 1$ (area A_2). Finally we subtract these two areas.

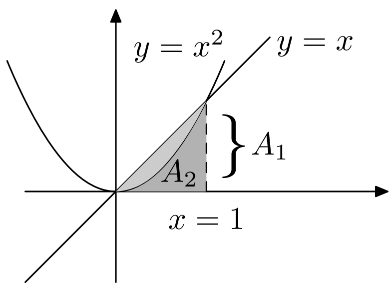

Figure 4.4: Area between $y = x^2$ and $y = x$= Area between $y = x$ and x-axis from $x = 0$ to $x = 1$ - Area between $y = x^2$ and x-axis from $x = 0$ to $x = 1$.

$$\text{Area} = A_1 - A_2 = \int_0^1 x\,dx - \int_0^1 x^2\,dx = \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Exercises 2.7

Q1 Find the area enclosed by $y = 2x$ and the x-axis from $x = -2$ to $x = 2$.

Q2 Find the area enclosed by $y = x^2 - 3x$ and the x-axis.

Q3 Find the area enclosed by the functions $y = x^2$ and $y = 2x$.

Q4 Find the area enclosed by the functions $y = x^2$ and $y = \sqrt{x}$.

Q5 Find the area enclosed by the functions $y = \sqrt{x}$ and $y = x$.

4.8 Volume

Consider the area between the x-axis from $x = a$ to $x = b$.

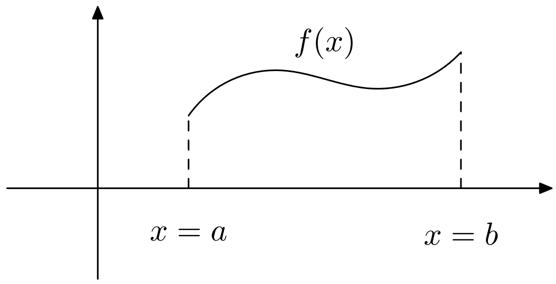

Figure 4.5: Area between the x-axis and a function from $x = a$ to $x = b$

Using integration, we can find the volume of the object obtained by rotating the above area about the x-axis by the formula $V = \pi \int_a^b (f(x))^2\,dx = \pi \int_a^b y^2\,dx$.

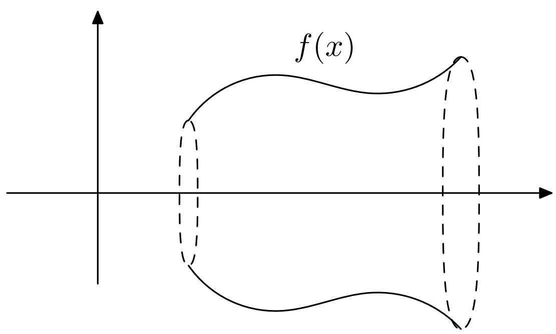

Figure 4.6: The above area rotated about the x-axis

Example 4.48 *Consider the area enclosed between $y = x^2 - x$ and the x-axis. Find the volume of the object obtained by rotating this area about the x-axis.*

We substitute the expression of $y = x^2 - x$ in the above formula and we obtain:

$$V = \pi \int_a^b y^2\,dx = \int_0^1 (x^2 - x)^2\,dx = \pi \int_0^1 (x^4 - 2x^3 + x^2)\,dx = \pi \left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]_0^1$$
$$= \pi \left[\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right] = \frac{\pi}{30}.$$

Example 4.49 *Consider the area enclosed between $y = x^2$ and $y = x$. Find the volume of the object obtained by rotating this area about the x-axis.*

We have already sketched the area enclosed between $y = x^2$ and $y = x$ (figure 2.4). We can't calculate this volume directly using integration. To find the volume of this object, we:

- Find the volume of the object obtained by rotating the area between $y = x$ and the x-axis from $x = 0$ to $x = 1$ about the x-axis (V_1).
- Find the volume of the object obtained by rotating the area between $y = x^2$ and the x-axis from $x = 0$ to $x = 1$ about the x-axis (V_2).
- Subtract these two volumes ($V_1 - V_2$).

$$V = V_1 - V_2 = \pi \int_0^1 (x)^2\,dx - \pi \int_0^1 (x^2)^2\,dx = \pi \int_0^1 x^2\,dx - \pi \int_0^1 x^4\,dx = \pi \left[\frac{x^3}{3}\right]_0^1 - \pi \left[\frac{x^5}{5}\right]_0^1 = \pi \left[\frac{1}{3} - \frac{1}{5}\right]$$
$$= \frac{2\pi}{15}.$$

Exercises 2.8

Q1 Consider the area enclosed by $y = 2x$ and the x-axis from $x = 0$ to $x = 2$. Find the volume of the object obtained by rotating this area about the x-axis.

Q2 Consider the area enclosed by $y = x^2 - 3x$ and the x-axis. Find the volume of the object obtained by rotating this area about the x-axis.

Q3 Consider the area enclosed by the functions $y = x^2$ and $y = 2x$. Find the volume of the object obtained by rotating this area about the x-axis.

Q4 Consider the area enclosed by the functions $y = x^2$ and $y = \sqrt{x}$. Find the volume of the object obtained by rotating this area about the x-axis.

Q5 Consider the area enclosed by the functions $y = \sqrt{x}$ and $y = x$. Find the volume of the object obtained by rotating this area about the x-axis.

4.9 Answers

Exercises 2.1.1

Q1 (i) $\frac{x^4}{4} - 7\cos x + 8\sin x + c$, (ii) $-\frac{7x^{-3}}{3} + \frac{7x^{-4}}{4} - 2\sqrt{x} + 7x + c$,
(iii) $-\frac{x^{-6}}{6} - 2\ln x + \frac{x^{12}}{12} + \frac{4x^{-3}}{3} + 21x + c$, (iv) $\frac{1}{6}$.

Exercises 2.1.2

Q1 (i) $\frac{91}{12}$, (ii) $3e + 10$, (iii) $\frac{15}{4} - \ln 16$, (iv) 2.

Exercises 2.2

Q1 (i) $-\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x$, (ii) $x^2\ln x - \frac{1}{2}x^2 + c$, (iii) $xe^x - e^x + c$, (iv) $\frac{1}{2}e^x(\sin x + \cos x) + c$,
(v) $x^2e^x - 2xe^x + 2e^x + c$, (vi) $-xe^{-x} - e^{-x} + c$.

Exercises 2.3

Q1 (i) $2\sqrt{x-3} + c$, (ii) $\frac{341}{5}$, (iii) $\frac{121}{38880}$, (iv) $\frac{\ln 2}{2}$, (v) $-\frac{1}{4}\ln(7-4x) + c$, (vi) $-\frac{1}{2}\cos(2x+5) + c$,
(vii) $\frac{2}{3}$, (viii) $\frac{1}{3}\sin(3x+11) + c$, (ix) $\frac{1}{2}$, (x) $\frac{e}{4} - \frac{1}{4}$, (xi) $\frac{1}{2}e^{2x+3} + c$.

Exercises 2.4

Q1 (i) $\frac{\ln|x-3| - \ln|x-1|}{2} + c$, (ii) $\frac{\ln 3 - \ln 4 + 2\ln 2}{2}$, (iii) $\frac{6\ln 6 - 6\ln 7 - 4\ln 2}{5}$.

Exercises 2.5

Q1 (i) $\frac{(x^4+9)^6}{6} + c$, (ii) $\frac{16}{3} - 2\sqrt{3}$, (iii) $\frac{3}{16}$, (iv) $\ln 2 - \frac{\ln 14}{4}$, (v) $\ln 9$, (vi) $-\cos(x^3+7) + c$,
(vii) $-\cos(x^2 + e^x) + c$, (viii) $\frac{1}{6}\sin(x^6+7) + c$, (ix) $\sin(x^3 + \ln x) + c$, (x) $\frac{1}{16}e^{4x^4} + c$, (xi) $e^{x+e^x} + c$.

Exercises 2.6

Q1 (i) $\frac{\pi}{2}$, (ii) $\frac{\pi}{16} - \frac{\sqrt{2}}{8} + \frac{1}{4}$, (iii) $\frac{4}{3}$, (iv) $\frac{2}{3}$, (v) 0, (vi) $\frac{1}{4}$.

Exercises 2.7

Q1 8. **Q2** $\frac{9}{2}$. **Q3** $\frac{4}{3}$. **Q4** $\frac{1}{3}$. **Q5** $\frac{1}{6}$.

Exercises 2.8

Q1 $\frac{32\pi}{3}$. **Q2** $\frac{81\pi}{10}$. **Q3** $\frac{16\pi}{15}$. **Q4** $\frac{9\pi}{70}$. **Q5** $\frac{\pi}{30}$.