University of Chester

Maths Notes for first year Engineers

3. Differentiation

Joe Gildea

Loukas Zagkos

Contents

Chapter 3

Differentiation

3.1 Limits

In this section we are going to be dealing with two types of limits:

Type 1 $\lim_{x\to a} f(x)$, i.e. Limits as $x \to a$ where a is a constant.

Type 2 $\lim_{x\to\infty} f(x)$, i.e. Limits as $x \to \infty$ where ∞ is infinity.

3.1.1 $\lim_{x\to a} f(x)$

The rules for evaluating these types of limits are:

- Simply replace x with a.
- If the expression obtained after this substitution is $\left(\frac{0}{0}\right)$, we must simplify it to determine the limit. These expressions are called indeterminate forms.

Let's look at a few examples.

Example 3.1 *Evaluate* $\lim_{x\to 2}(2x-3)$.

In this example $f(x) = 2x - 3$. We simply replace x with 2.

$$\lim_{x\to 2}(2x-3) = 2\cdot 2 - 3 = 1.$$

Example 3.2 *Evaluate* $\lim_{x\to 1}\frac{x-1}{x^2-1}$.

Let's replace x with 1 and see what happens.

$$\lim_{x\to 1}\frac{x-1}{x^2-1} = \frac{1-1}{1^2-1} = \frac{0}{0}.$$

We need to simplify this expression. $f(x) = \dfrac{x-1}{x^2-1} = \dfrac{x-1}{(x-1)(x+1)} = \dfrac{1}{x+1}$. Therefore,

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2}.$$

Here's a few important points:

- First replace x with a. If the obtained expression is $\left(\dfrac{0}{0}\right)$, go back and simplify it.
- If the limit is ∞, we say the limit doesn't exist.

Example 3.3 *Evaluate* $\lim\limits_{x\to 1} \dfrac{1}{x^2-1}$.

$$\lim_{x\to 1} \frac{1}{x^2-1} = \frac{1}{1^2-1} = \frac{1}{0} = \infty, \quad \text{i.e. the limit doesn't exist.}$$

Example 3.4 *Evaluate* $\lim\limits_{x\to 1} \dfrac{x^2-1}{x^2+x-2}$.

Let's replace x with 1.

$$\lim_{x\to 1} \frac{x^2-1}{x^2+x-2} = \frac{1^2-1}{1^2+1-2} = \frac{0}{0}.$$

We need to simplify this expression. $f(x) = \dfrac{x^2-1}{x^2+x-2} = \dfrac{(x-1)(x+1)}{(x-1)(x+2)} = \dfrac{(x+1)}{(x+2)}$.

Therefore, $\quad \lim\limits_{x\to 1} \dfrac{x^2-1}{x^2-2x+1} = \lim\limits_{x\to 1} \dfrac{(x+1)}{(x+2)} = \dfrac{1+1}{1+2} = \dfrac{2}{3}$.

Exercises 1.1.1

Q1 Evaluate the following limits:

(i) $\lim\limits_{x\to 1} (x+4)$.

(ii) $\lim\limits_{x\to 3} (5x-7)$.

(iii) $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-4}$.

(iv) $\lim\limits_{x\to 3} \dfrac{x-3}{x^2-9}$.

(v) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-x-2}$.

(vi) $\lim\limits_{x\to 3} \dfrac{x^2-9}{x^2-2x-3}$.

3.1.2 Trigonometric Limits

Before attempting any trigonometric limits, we have to observe the following rules concerning $\sin\theta$ and $\cos\theta$:

- $\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = \lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = 1.$
- $\lim_{\theta\to 0}\left(\frac{\cos\theta}{1}\right) = \lim_{\theta\to 0}\left(\frac{1}{\cos\theta}\right) = 1.$
- $\lim_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right) = \lim_{\theta\to 0}\left(\frac{\theta}{\tan\theta}\right) = 1.$

Now we can attempt the following examples.

Example 3.5 *Evaluate* $\lim_{\theta\to 0}\left(\frac{\sin(6\theta)}{(5\theta)}\right)$.

$$\lim_{\theta\to 0}\left(\frac{\sin(6\theta)}{(5\theta)}\right) = \lim_{\theta\to 0}\left[\left(\frac{\sin(6\theta)}{(6\theta)}\right)\left(\frac{6\theta}{5\theta}\right)\right] = \lim_{\theta\to 0}\left[\left(\frac{\sin(6\theta)}{(6\theta)}\right)\left(\frac{6}{5}\right)\right] = 1\cdot\frac{6}{5} = \frac{6}{5}.$$

Example 3.6 *Evaluate* $\lim_{x\to 0}\left(\frac{\sin(3x)}{\sin(8x)}\right)$.

$$\lim_{x\to 0}\left(\frac{\sin(3x)}{\sin(8x)}\right) = \lim_{x\to 0}\left[\left(\frac{\sin(3x)}{3x}\right)\left(\frac{8x}{\sin(8x)}\right)\left(\frac{3x}{8x}\right)\right] = 1\cdot 1\cdot\frac{3}{8} = \frac{3}{8}.$$

Example 3.7 *Evaluate* $\lim_{\theta\to 0}\left(\frac{\tan(3\theta)}{\cos(4\theta)\sin(5\theta)}\right)$.

$$\lim_{\theta\to 0}\left(\frac{\tan(3\theta)}{\cos(4\theta)\sin(5\theta)}\right) = \lim_{\theta\to 0}\left[\left(\frac{\tan(3\theta)}{3\theta}\right)\left(\frac{1}{\cos(4\theta)}\right)\left(\frac{5\theta}{\sin(5\theta)}\right)\left(\frac{3\theta}{5\theta}\right)\right] = 1\cdot 1\cdot 1\cdot\frac{3}{5} = \frac{3}{5}.$$

Example 3.8 *Evaluate* $\lim_{x\to 0}\left(\frac{\sin(3x^2)}{\sin^2(x)}\right)$.

$$\lim_{x\to 0}\left(\frac{\sin(3x^2)}{\sin^2(x)}\right) = \lim_{x\to 0}\left(\frac{\sin(3x^2)}{\sin(x)\sin(x)}\right) = \lim_{x\to 0}\left[\left(\frac{\sin(3x^2)}{3x^2}\right)\left(\frac{x}{\sin(x)}\right)\left(\frac{x}{\sin(x)}\right)\left(\frac{\sin(3x^2)}{x^2}\right)\right] = 1{\cdot}1{\cdot}1{\cdot}3 = 3.$$

Example 3.9 *Evaluate* $\lim_{\theta\to 0}\left(\frac{\sin(2\theta)\tan(5\theta)}{\theta^2\cos(4\theta)}\right)$.

$$\lim_{\theta\to 0}\left(\frac{\sin(2\theta)\tan(5\theta)}{\theta^2\cos(4\theta)}\right) = \lim_{\theta\to 0}\left[\left(\frac{\sin(2\theta)}{2\theta}\right)\left(\frac{\tan(5\theta)}{5\theta}\right)\left(\frac{1}{\cos(4\theta)}\right)\left(\frac{10\theta^2}{\theta^2}\right)\right] = 1\cdot 1\cdot 1\cdot 10 = 10.$$

Exercises 1.1.2

Q1 Evaluate the following limits:

(i) $\lim_{\theta \to 0} \dfrac{\sin(7\theta)}{\theta}$.

(ii) $\lim_{\theta \to 0} \dfrac{5\theta}{\tan(\theta)}$.

(iii) $\lim_{\theta \to 0} \dfrac{\sin(3\theta)}{\tan(4\theta)}$.

(iv) $\lim_{\theta \to 0} \dfrac{\sin(5\theta)}{\cos(3\theta)\sin(3\theta)}$.

3.1.3 $\lim_{x \to \infty} f(x)$

Consider $f(x) = \left(\dfrac{1}{x}\right)$. Let's see what happens to $\left(\dfrac{1}{x}\right)$ as x increases.

$$\begin{array}{c|c} x & \left(\dfrac{1}{x}\right) \\ \hline\hline 1 & \dfrac{1}{1} = 1 \\ 2 & \dfrac{1}{2} = 0.5 \\ \vdots & \vdots \\ 10 & \dfrac{1}{10} = 0.1 \\ \vdots & \vdots \\ 100 & \dfrac{1}{100} = 0.01 \\ \vdots & \vdots \\ 1000 & \dfrac{1}{1000} = 0.001 \end{array} \tag{3.1}$$

We can see that as x increases $\left(\dfrac{1}{x}\right)$ get's smaller. Thus if x get's infinitely large, then $\left(\dfrac{1}{x}\right)$ approaches 0, i.e.

$$\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0.$$

We can extend this result to

$$\lim_{x \to \infty} \left(\frac{1}{x^p}\right) = 0 \quad \text{when } p > 0.$$

Now we use this fact in evaluating $\lim_{x \to \infty} f(x)$.

Example 3.10 *Evaluate* $\lim_{x\to\infty} \dfrac{x^2+x+1}{x^2+3x+5}$.

To evaluate this limit, we divide up and below by x^k (where k is the highest power of x) and then use the fact that $\lim_{x\to\infty}\left(\dfrac{1}{x}\right) = 0$. Here $k = 2$. Thus we divide up and below by x^2.

$$\lim_{x\to\infty} \frac{x^2+x+1}{x^2+3x+5} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{3x}{x^2}+\frac{5}{x^2}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{1}{x^2}}{1+3\left(\frac{1}{x}\right)+5\left(\frac{1}{x^2}\right)}.$$

Now we use the fact that $\lim_{x\to\infty}\left(\dfrac{1}{x^p}\right) = 0$ $p > 0$. i.e. $\lim_{x\to\infty} \dfrac{x^2+x+1}{x^2+3x+5} = \dfrac{1+0+0}{1+0+0} = 1$.

Example 3.11 *Evaluate* $\lim_{x\to\infty} \dfrac{x^3+x+4}{2x^3+3x^2+2}$.

Here we divide up and below by x^3 and then use the fact that $\lim_{x\to\infty}\left(\dfrac{1}{x^p}\right) = 0$ $(p > 0)$ and we get

$$\lim_{x\to\infty} \frac{x^3+x+4}{2x^3+3x^2+2} = \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{x}{x^3}+\frac{4}{x^3}}{\frac{2x^3}{x^3}+\frac{3x^2}{x^3}+\frac{2}{x^3}} = \lim_{x\to\infty} \frac{1+\frac{1}{x^2}+\frac{4}{x^3}}{2+\frac{3}{x}+\frac{2}{x^3}} = \frac{1+0+0}{2+0+0} = \frac{1}{2}.$$

Exercises 1.1.3

Q1 Evaluate the following limits:

(i) $\lim_{x\to\infty} \dfrac{2x^2+2x-3}{x^2-x-1}$.

(ii) $\lim_{x\to\infty} \dfrac{x^3+2x-3}{5x^2+3x+12}$.

(iii) $\lim_{x\to\infty} \dfrac{x^3+x^2+x+4}{3x^2+5x+2}$.

(iv) $\lim_{x\to\infty} \dfrac{x^7+x^2+4}{2x^5+5x+2}$.

(v) $\lim_{x\to\infty} \dfrac{x^7+x^2+4}{2x^7+5x+2}$.

(vi) $\lim_{x\to\infty} \dfrac{x^5+x^2+4}{2x^7+5x+2}$.

3.2 Differentiation From 1^{st} Principles

The first derivative of a function from 1^{st} principles is :

$$\frac{dy}{dx} \text{ (or } f'(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Here's a few important points. When asked to differentiate $f(x)$ from 1^{st} principles:

- Substitute the expressions of $f(x)$ and $f(x+h)$ in $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.
- Replace h with 0 and evaluate the limit.
- If the limit is $\left(\frac{0}{0}\right)$, simplify it first and then evaluate.

Example 3.12 *Differentiate $f(x) = 3x - 5$ from 1^{st} principles.*

Let $f(x) = 3x - 5$, then $f(x+h) = 3(x+h) - 5 = 3x + 3h - 5$.

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{3x+3h-5-(3x-5)}{h} \\
&= \lim_{h\to 0} \frac{3x+3h-5-3x+5}{h} \\
&= \lim_{h\to 0} \frac{3h}{h} \\
&= \lim_{h\to 0} 3 \\
&= 3.
\end{aligned}$$

Example 3.13 *Differentiate $f(x) = x^2 - x + 1$ from 1^{st} principles.*

Let $f(x) = x^2 - x + 1$, then $f(x+h) = (x+h)^2 - (x+h) + 1 = x^2 + 2xh + h^2 - (x+h) + 1 = x^2 + 2xh + h^2 - x - h + 1$.

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2-x-h+1-(x^2-x+1)}{h} \\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2-x-h+1-x^2+x-1}{h} \\
&= \lim_{h\to 0}\frac{2xh+h^2-h}{h} \\
&= \lim_{h\to 0}\frac{h(2x+h-1)}{h} \\
&= \lim_{h\to 0} 2x+h-1 \\
&= 2x+0-1. \\
&= 2x-1.
\end{aligned}
$$

Example 3.14 *Differentiate $f(x) = x^3$ from 1^{st} principles.*

Let $f(x) = x^3$, then $f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$.

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \\
&= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} \\
&= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h} \\
&= \lim_{h\to 0} 3x^2+3xh+h^2 \\
&= 3x^2+0+0. \\
&= 3x^2.
\end{aligned}
$$

Exercises 1.2

Q1 Differentiate each of the following from 1^{st} principles:
(i) $y = x$.
(ii) $y = 4x$.

(iii) $y = x - 3$.
(iv) $y = 3x + 2$.
(v) $y = 2x^2$.
(vi) $y = 2x^2 - x$.

3.3 Differentiation of $\sin(x)$ and $\cos(x)$ from 1^{st} principles

Example 3.15 *Differentiate* $\sin(x)$ *from* 1^{st} *principles.*

Let $f(x) = \sin(x)$, then $f(x+h) = \sin(x+h)$.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}.\end{aligned}$$

Now we have to apply the trigonometric identity:

$$\sin(A) - \sin(B) = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right).$$

Let $A = (x+h)$ and $B = (x)$, then

- $\dfrac{x-y}{2} = \dfrac{x+h-x}{2} = \dfrac{h}{2}$.
- $\dfrac{x+y}{2} = \dfrac{x+h+x}{2} = \dfrac{2x+h}{2} = x + \dfrac{h}{2}$.

Therefore

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{2\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)}{h}\\ &= \lim_{h\to 0} 2\left(\frac{\sin\left(\frac{h}{2}\right)}{2\left(\frac{h}{2}\right)}\right)\left(\cos\left(x+\frac{h}{2}\right)\right)\\ &= \lim_{h\to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right)\left(\cos\left(x+\frac{h}{2}\right)\right)\\ &= 1\cdot(\cos(x+0))\\ &= \cos(x).\end{aligned}$$

Example 3.16 *Differentiate* $\cos(x)$ *from* 1^{st} *principles.*

Let $f(x) = \cos(x)$, then $f(x+h) = \cos(x+h)$.

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}.
\end{aligned}
$$

Now we have to apply the trigonometric identity:

$$\cos(A) - \cos(B) = -2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right).$$

Let $A = (x+h)$ and $B = (x)$, then

- $\dfrac{x-y}{2} = \dfrac{x+h-x}{2} = \dfrac{h}{2}.$
- $\dfrac{x+y}{2} = \dfrac{x+h+x}{2} = \dfrac{2x+h}{2} = x + \dfrac{h}{2}.$

Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{-2\sin\left(\frac{h}{2}\right)\sin\left(x + \frac{h}{2}\right)}{h} \\
&= \lim_{h \to 0} -2\left(\frac{\sin\left(\frac{h}{2}\right)}{2\left(\frac{h}{2}\right)}\right)\left(\sin\left(x + \frac{h}{2}\right)\right) \\
&= \lim_{h \to 0} \left(-\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right)\left(\sin\left(x + \frac{h}{2}\right)\right) \\
&= -1 \cdot (\sin(x+0)) \\
&= -\sin(x).
\end{aligned}
$$

3.4 Derivative of Basic Functions

Here's a few basic rules of differentiation.

Rule (1a): If $y = k$, where k is a constant, then $\dfrac{dy}{dx} = 0$.

Example 3.17 *If* $y = 3$, *find* $\dfrac{dy}{dx}$.

$\dfrac{dy}{dx} = 0.$

Rule (1b): If $y = x^n$, then $\dfrac{dy}{dx} = n \cdot x^{n-1}$.

Example 3.18 *If* $y = x^5$*, find* $\dfrac{dy}{dx}$.

$\dfrac{dy}{dx} = 5x^4.$

Note that $\dfrac{dy}{dx}$ and $f'(x)$ are the same.

Example 3.19 *If* $f(x) = x^2$*, find* $f'(x)$.

$f'(x) = \dfrac{dy}{dx} = 2x^1 = 2x.$

Example 3.20 *If* $y = \dfrac{1}{x^2}$*, find* $\dfrac{dy}{dx}$.

First we will have to bring the x^2 above the line and rewrite y as $y = x^{-2}$.

$\dfrac{dy}{dx} = -2x^{-3} = -2\dfrac{1}{x^3} = \dfrac{-2}{x^3}.$

Example 3.21 *If* $y = \sqrt{x}$*, find* $\dfrac{dy}{dx}$.

First we will have to rewrite y as $y = x^{\frac{1}{2}}$.

$\dfrac{dy}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2}\dfrac{1}{x^{\frac{1}{2}}} = \dfrac{1}{2\sqrt{x}}.$

Example 3.22 *If* $y = \dfrac{1}{\sqrt{x}}$*, find* $\dfrac{dy}{dx}$.

First we will have to rewrite y as $y = x^{-\frac{1}{2}}$.

$\dfrac{dy}{dx} = -\dfrac{1}{2}x^{-\frac{3}{2}} = \dfrac{1}{2}\dfrac{1}{x^{\frac{3}{2}}} = -\dfrac{1}{2(\sqrt{x})^3}.$

Rule (1c): If $y = a \cdot x^n$, then $\dfrac{dy}{dx} = a \cdot n \cdot x^{n-1}$.

Example 3.23 *If* $y = 3x^3$*, find* $\dfrac{dy}{dx}$.

In this question we differentiate the x^3 and then multiply the answer by 3.

$\dfrac{dy}{dx} = 3 \cdot 3x^2 = 9x^2.$

Example 3.24 *If* $y = 3\sqrt{x}$*, find* $\dfrac{dy}{dx}$.

First we will have to bring the $\sqrt{x}$ above the line and rewrite y as $y = 3x^{\frac{1}{2}}$.

$\dfrac{dy}{dx} = 3\dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{3}{2}\dfrac{1}{x^{\frac{1}{2}}} = \dfrac{3}{2\sqrt{x}}.$

Rule (2a): If $y = \sin x$, then $\dfrac{dy}{dx} = \cos x$.

$$\boxed{\textbf{Rule (2b):} \text{ If } y = a \cdot \sin x \text{ then } \frac{dy}{dx} = a \cdot \cos x.}$$

Example 3.25 *If $y = 10 \sin x$, find* $\dfrac{dy}{dx}$.

$\dfrac{dy}{dx} = 10 \cos x.$

$$\boxed{\textbf{Rule (3a):} \text{ If } y = \cos x \text{ then } \frac{dy}{dx} = -\sin x.}$$

$$\boxed{\textbf{Rule (3b):} \text{ If } y = a \cdot \cos x \text{ then } \frac{dy}{dx} = -a \cdot \sin x.}$$

Example 3.26 *If $y = 15 \cos x$, find* $\dfrac{dy}{dx}$.

$\dfrac{dy}{dx} = -15 \sin x.$

$$\boxed{\textbf{Rule (4a):} \text{ If } y = e^x \text{ then } \frac{dy}{dx} = e^x.}$$

$$\boxed{\textbf{Rule (4b):} \text{ If } y = a \cdot e^x \text{ then } \frac{dy}{dx} = a \cdot e^x.}$$

Example 3.27 *If $y = 7e^x$, find* $\dfrac{dy}{dx}$.

$\dfrac{dy}{dx} = 7e^x.$

$$\boxed{\textbf{Rule (5a):} \text{ If } y = \ln x \text{ then } \frac{dy}{dx} = \frac{1}{x}.}$$

$$\boxed{\textbf{Rule (5b):} \text{ If } y = a \cdot \ln x \text{ then } \frac{dy}{dx} = a \cdot \frac{1}{x} = \frac{a}{x}.}$$

Example 3.28 *If $y = 4 \ln x$, find* $\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{4}{x}.$

3.5 Basic Rules of Differentiation

When we are differentiating the addition/subtraction of functions, we differentiate the functions individually and then add/subtract them. Therefore, if $y(x) = u(x) \pm v(x)$, then

$$\boxed{y'(x) = u'(x) \pm v'(x).}$$

Example 3.29 *If* $y = x^2 + \sin x + \frac{1}{x} + 7\ln x$, *find* $\frac{dy}{dx}$.

First rewrite y as $y = x^2 + \sin x + x^{-1} + 7\ln x$.

$$\frac{dy}{dx} = 2x + \cos x - 1x^{-2} + \frac{7}{x} = 2x + \cos x - \frac{1}{x^2} + \frac{7}{x}.$$

Example 3.30 *If* $f(x) = \cos x + \frac{1}{\sqrt[3]{x}} + e^x$, *find* $f'(x)$.

First rewrite $f(x)$ as $f(x) = \cos x + x^{-\frac{1}{3}} + e^x$.

$$f'(x) = -\sin x - \frac{1}{3}x^{-\frac{4}{3}} + e^x = -\sin x - \frac{1}{3(\sqrt[3]{x})^4} + e^x.$$

Exercises 11.2

Q1 Find $\frac{dy}{dx}$ of the following functions:

(i) $y = x^7 - 5x + 3 + \frac{1}{x^4}$.

(ii) $y = x^5 + 3\cos x + 4\sin x - 5e^x$.

(iii) $y = x^8 + 7\ln x - 7$.

(iv) $y = \frac{4}{x^3} - \frac{1}{x^4} + \frac{3}{x^5}$.

(v) $y = \frac{1}{\sqrt{x}} + \frac{2}{\sqrt[3]{x}} - \frac{1}{\sqrt[4]{x}}$.

3.5.1 The Product Rule

The product rule is :

$$\boxed{\text{If } y = u \cdot v, \text{ then } \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.}$$

i.e. if $f(x)$ is the product of two functions (1^{st} function $\times$ 2^{nd} function), then

$$\frac{dy}{dx} = 1^{\text{st}} \text{ function} \times \begin{array}{c}\text{The derivative of}\\ 2^{\text{nd}} \text{ function}\end{array} + 2^{\text{nd}} \text{ function} \times \begin{array}{c}\text{The derivative of}\\ 1^{\text{st}} \text{ function}\end{array}$$

Example 3.31 *If* $y = (x^2 - x)(x^7 + x^3 + 7)$, *find* $\dfrac{dy}{dx}$.

Let $u = x^2 - x$ (1$^{\text{st}}$ function) and let $v = x^7 + x^3 + 7$ (2$^{\text{nd}}$ function). Then

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= (x^2 - x)(7x^6 + 3x^2) + (x^7 + x^3 + 7)(2x - 1).\end{aligned}$$

Example 3.32 *If* $y = (x^2 - x)\sin x$, *find* $\dfrac{dy}{dx}$.

Let $u = x^2 - x$ (1$^{\text{st}}$ function) and let $v = \sin x$ (2$^{\text{nd}}$ function). Then

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= (x^2 - x)(\cos x) + (\sin x)(2x - 1).\end{aligned}$$

Example 3.33 *If* $y = (x^2)e^x$, *find* $\dfrac{dy}{dx}$.

Let $u = x^2$ (1$^{\text{st}}$ function) and let $v = e^x$ (2$^{\text{nd}}$ function). Then

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= (x^2)(e^x) + (e^x)(2x)\\ &= e^x(x^2 + 2x).\end{aligned}$$

Example 3.34 *If* $y = (3x^3 + 7)\ln x$, *find* $\dfrac{dy}{dx}$.

Let $u = 3x^3 + 7$ (1$^{\text{st}}$ function) and let $v = \ln x$ (2$^{\text{nd}}$ function). Then

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= (3x^3 + 7)(\frac{1}{x}) + (\ln x)(9x^2)\\ &= \left(\frac{3x^3 + 7}{x}\right) + 9x^2\ln x.\end{aligned}$$

Example 3.35 *If* $y = \sin x \ln x$, *find* $\dfrac{dy}{dx}$.

Let $u = \sin x$ (1$^{\text{st}}$ function) and let $v = \ln x$ (2$^{\text{nd}}$ function). Then

$$\begin{aligned}
\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\
&= (\sin x)(\frac{1}{x}) + (\ln x)(\cos x) \\
&= \left(\frac{\sin x}{x}\right) + \cos x \ln x.
\end{aligned}$$

Example 3.36 *If $y = e^x \cos x$, find $\frac{dy}{dx}$.*

Let $u = e^x$ (1$^{\text{st}}$ function) and let $v = \cos x$ (2$^{\text{nd}}$ function). Then

$$\begin{aligned}
\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\
&= (e^x)(-\sin x) + (\cos x)(e^x) \\
&= -e^x \sin x + e^x \cos x.
\end{aligned}$$

Example 3.37 *If $y = \sin x \cos x$, find $\frac{dy}{dx}$.*

Let $u = \sin x$ (1$^{\text{st}}$ function) and let $v = \cos x$ (2$^{\text{nd}}$ function). Then

$$\begin{aligned}
\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\
&= (\sin x)(-\sin x) + (\cos x)(\cos x) \\
&= -\sin^2 x + \cos^2 x \\
&= \cos^2 x - \sin^2 x \\
&= \cos(2x).
\end{aligned}$$

Exercises 1.5.1

Q1 Find $\frac{dy}{dx}$ of the following functions:

(i) $y = (7x^5 + 3x)\left(x^2 - x + \frac{1}{x}\right)$.
(ii) $y = \left(\frac{1}{x} - \frac{1}{x^2}\right)(x^2 + 3x)$.
(iii) $y = (x^4 + x^3)\sin x$.
(iv) $y = x^3 e^x$.
(v) $y = e^x \ln x$.
(vi) $y = e^x \sin x$.

(vii) $y = \sin x \ln x$.

(viii) $y = \left(x^2 + x + \dfrac{1}{x}\right) \ln x$.

3.5.2 The Quotient Rule

The Quotient rule is :

$$\boxed{\text{If } y = \frac{u}{v}, \text{ then } \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.}$$

Example 3.38 *If* $y = \dfrac{(x^2+7x+3)}{(x^2+3)}$, *find* $\dfrac{dy}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\
&= \frac{(x^2+3)(2x+7) - (x^2+7x+3)(2x)}{(x^2+3)^2} \\
&= \frac{2x^3+7x^2+6x+21 - \{2x^3+14x^2+6x\}}{(x^2+3)^2} \\
&= \frac{2x^3+7x^2+6x+21-2x^3-14x^2-6x}{(x^2+3)^2} \\
&= \frac{-7x^2+21}{(x^2+3)^2}.
\end{aligned}$$

Example 3.39 *If* $y = \dfrac{\sin x}{e^x}$, *find* $\dfrac{dy}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\
&= \frac{(e^x)(\cos x) - (\sin x)(e^x)}{(e^x)^2} \\
&= \frac{e^x\cos x - e^x \sin x}{e^{2x}}.
\end{aligned}$$

Example 3.40 *If* $y = \dfrac{\sin x}{\ln x}$, *find* $\dfrac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{(\ln x)(\cos x) - (\sin x)\left(\frac{1}{x}\right)}{(\ln x)^2} \\ &= \frac{\cos x \ln x - \left(\frac{\sin x}{x}\right)}{(\ln x)^2}.\end{aligned}$$

Exercises 1.5.2

Q1 Find $\frac{dy}{dx}$ of the following functions:

(i) $y = \dfrac{x+3}{x+1}$.

(ii) $y = \dfrac{x^2 - 3x + 1}{x^2 + 1}$.

(iii) $y = \dfrac{\cos x}{e^x}$.

(iv) $y = \dfrac{\ln x}{\cos x}$.

3.5.3 The Chain Rule

First let's remind ourselves of some basic rules of differentiation.

y **or** $f(x)$	$\frac{dy}{dx}$ **or** $f'(x)$
x^n	$n \cdot x^{n-1}$
k	0
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

The question that arises is, what if there was a more complex function and we wanted to differentiate it. We could not use any of the basic rules listed above. In this case we would need the chain rule:

$$\boxed{\text{If } y = f(g(x)), \text{ then } \frac{dy}{dt} = f'(g(x)) \cdot g'(x).}$$

Example 3.41 *If* $y = (x^2 + 2)^3$, *find* $\dfrac{dy}{dx}$.

It is clear at this point that we can't use one of the basic rules to differentiate y. Thus we have to use the chain rule. Here's a few steps to apply the chain rule:

(1) Identify a basic rule from the given question. In the above example, we would identify the x^n as the basic rule. i.e. $y = (\blacksquare)^3$, where $\blacksquare = x^2 + 2$.

(2) Look at the basic rule and the given question and determine what x has been extended to. In the above question, the basic rule is x^n and the question is $y = (x^2 + 2)^3$, thus x has been extended to $x^2 + 2$. In other words, $g(x) = x^2 + 2$.

(3) The chain rule says differentiate using the simple rule anyway, however multiply by the derivative of what x has been extended to $(g(x))$. In this question $y = (x^2 + 2)^3$. Now when we use the simple rule, we get $f'(g(x)) = 3(x^2 + 2)^2$. The derivative of $g(x)$ is $g'(x) = 2x$. Therefore

$$\therefore \quad \frac{dy}{dx} = 3(x^2 + 2)^2 \cdot \frac{d}{dx}(x^2 + 2) = 3(x^2 + 2)^2(2x) = 6x(x^2 + 2)^2$$

where $\dfrac{d}{dx}(x^2 + 2)$ is the derivative of $x^2 + 2$.

Example 3.42 *If* $y = (x^7 + x + 7)^{90}$, *find* $\dfrac{dy}{dx}$.

It is clear at this point that we can't use one of the basic rules to differentiate y. Thus we have to use the chain rule.

(1) In this example, we would identify the x^n as the basic rule. i.e. $y = (\blacksquare)^{90}$, where $\blacksquare = x^7 + x + 7$.

(2) In this question x has been extended to $x^7 + x + 7$, thus $g(x) = x^7 + x + 7$.

(3) In this question $y = (x^7 + x + 7)^{90}$. Now when we use the simple rule, we get $f'(g(x)) = 90(x^7 + x + 7)^{89}$. The derivative of $g(x)$ is $g'(x) = 7x^6 + 1$. Therefore

$$\therefore \quad \frac{dy}{dx} = 90(x^7 + x + 7)^{89} \cdot \frac{d}{dx}(x^7 + x + 7) = 90(x^7 + x + 7)^{89}(7x^6 + 1).$$

Example 3.43 *If* $y = \dfrac{1}{(x^3 + x)^3}$, *find* $\dfrac{dy}{dx}$.

First we rewrite y as $y = (x^3 + x)^{-3}$. It is clear at this point that we can't use one of the basic rules to differentiate y. Thus we have to use the chain rule.

(1) In this example, we would identify the x^n as the basic rule. i.e. $y = (\blacksquare)^{-3}$, where $\blacksquare = x^3 + x$.

(2) In this question $g(x) = x^3 + x$.

(3) In this question $y = (x^3+x)^{-3}$. Now when we use the simple rule, we get $f'(g(x)) = -3(x^3+x)^{-4}$. The derivative of $g(x)$ is $g'(x) = 3x^2 + 1$. Therefore

$$\therefore \quad \frac{dy}{dx} = -3(x^3+x)^{-4} \cdot \frac{d}{dx}(x^3+x) = -3(x^3+x)^{-4}(3x^2+1) = \frac{-3(3x^2+1)}{(x^3+x)^4}.$$

Example 3.44 *If* $y = \sin(2x)$, *find* $\frac{dy}{dx}$.

It is clear at this point that we can't use one of the basic rules to differentiate y. Thus we have to use the chain rule.

(1) In this example, we would identify the $\sin x$ as the basic rule. i.e. $y = \sin(\blacksquare)$, where $\blacksquare = 2x$.

(2) In this question $g(x) = 2x$.

(3) In this question $y = \sin(2x)$. Now when we use the simple rule, we get $f'(g(x)) = \cos(2x)$. The derivative of $g(x)$ is $g'(x) = 2x$. Therefore

$$\therefore \quad \frac{dy}{dx} = \cos(2x) \cdot \frac{d}{dx}(2x) = \cos(2x)(2) = 2\cos(2x).$$

Example 3.45 *If* $y = \cos(x^4+4)$, *find* $\frac{dy}{dx}$.

It is clear at this point that we can't use one of the basic rules to differentiate y. Thus we have to use the chain rule.

(1) In this example, we would identify the $\cos x$ as the basic rule. i.e. $y = \cos(\blacksquare)$, where $\blacksquare = x^4+4$.

(2) In this question $g(x) = x^4 + 4$.

(3) In this question $y = \cos(x^4+4)$. Now when we use the simple rule, we get $f'(g(x)) = -\sin(x^4+4)$. The derivative of $g(x)$ is $g'(x) = 4x^3$. Therefore

$$\therefore \quad \frac{dy}{dx} = -\sin(x^4+4) \cdot \frac{d}{dx}(x^4+4) = -\sin(x^4+4)(4x^3) = -4x^3\sin(x^4+4).$$

Example 3.46 *If* $y = \ln(x^2+x+7)$, *find* $\frac{dy}{dx}$.

It is clear at this point that we can't use one of the basic rules to differentiate y. Thus we have to use the chain rule.

(1) In this example, we would identify the $\ln x$ as the basic rule. i.e. $y = \ln(\blacksquare)$, where $\blacksquare = x^2+x+7$.

(2) In this question $g(x) = x^2 + x + 7$.

(3) In this question $y = \ln x^2 + x + 7$. Now when we use the simple rule, we get $f'(g(x)) = \dfrac{1}{(x^2+x+7)}$. The derivative of $g(x)$ is $g'(x) = 2x + 1$. Therefore

$$\therefore \quad \frac{dy}{dx} = \frac{1}{(x^2+x+7)} \cdot \frac{d}{dx}(x^2+x+7) = \frac{1}{(x^2+x+7)}(2x+1) = \frac{2x+1}{x^2+x+7}.$$

Example 3.47 *If $y = e^{7x^5}$, find $\dfrac{dy}{dx}$.*

It is clear at this point that we can't use one of the basic rules to differentiate y. Thus we have to use the chain rule.

(1) In this example, we would identify the e^x as the basic rule. i.e. $y = e^{(\blacksquare)}$, where $\blacksquare = 7x^5$.

(2) In this question $g(x) = 7x^5$.

(3) In this question $y = e^{7x^5}$. Now when we use the simple rule, we get $f'(g(x)) = e^{7x^5}$. The derivative of $g(x)$ is $g'(x) = 35x^4$. Therefore

$$\therefore \quad \frac{dy}{dx} = e^{7x^5} \cdot \frac{d}{dx}(7x^5) = e^{7x^5}(35x^4) = 35x^4 e^{7x^5}.$$

Example 3.48 *If $y = \sin^2(2x^2+3)$, find $\dfrac{dy}{dx}$.*

Here is an example of a question where you have to use the chain rule twice. First rewrite y as $y = (\sin(2x^2+3))^2$ It is clear at this point that we can't use one of the basic rules to differentiate y. Thus we have to use the chain rule.

(1) In this example, we would identify the x^n as the basic rule. i.e. $y = (\blacksquare)^2$, where $\blacksquare = \sin(2x^2+3)$.

(2) In this question $g(x) = \sin(2x^2+3)$.

(3) In this question $y = (\sin(2x^2+3))^2$. Now when we use the simple rule, we get $f'(g(x)) = 2(\sin(2x^2+3))^1$. Therefore

$$\therefore \quad \frac{dy}{dx} = 2(\sin(2x^2+3))^1 \cdot \frac{d}{dx}(\sin(2x^2+3)).$$

Clearly we have to apply the chain rule to calculate $\dfrac{d}{dx}(\sin(2x^2+3))$.

$$\frac{d}{dx}(\sin(2x^2+3)) = \cos(2x^2+3)\frac{d}{dx}(2x^2+3).$$

Therefore

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= 2(\sin(2x^2+3))(\cos(2x^2+3))\frac{d}{dx}(2x^2+3) \\ &= 2(\sin(2x^2+3))(\cos(2x^2+3))(4x) \\ &= 8x\sin(2x^2+3)\cos(2x^2+3). \end{aligned}$$

Exercises 1.5.3

Q1 Find $\dfrac{dy}{dx}$ of the following functions:

(i) $y = (x^7 + 7x^3)^3$.
(ii) $y = (x^3 + x^2 + x + 7)^{10}$.
(iii) $y = \dfrac{1}{(x^2 + x + 2)^4}$.
(iv) $y = \sin 4x$.
(v) $y = \cos\,(x^3 + x^2 + x)$.
(vi) $y = \ln(x^2 + 2x)$.
(vii) $y = e^{4x^5+x+1}$.
(viii) $y = \cos^3\,(2x^2 + x + 7)$.

3.6 Implicit Differentiation

Normally we are given $y = f(x)$ (i.e. y in terms of x) and we then find $\dfrac{dy}{dx}$ using the rules from before. However there exists functions where we can't express y in terms of x. For example if we take the function $x^2 + xy + y^2 = 13$, we can't manipulate this formula for y. In these situations we use Implicit Differentiation to find $\dfrac{dy}{dx}$. The rules for Implicit Differentiation are :

(1) Differentiate both x and y the same way on both sides of the equal sign.
(2) When you differentiate y, multiply by $\dfrac{dy}{dx}$.

Example 3.49 *If $x^2 + xy + y^2 = 13$, find $\dfrac{dy}{dx}$.*

$$x^2 + xy + y^2 = 13$$
$$2x + ((x)(1)\left(\frac{dy}{dx}\right) + (y)(1)) + 2y\left(\frac{dy}{dx}\right) = 0.$$

When we differentiate x^2, we get $2x$. When we differentiate xy, we have to use the product rule to get $((x)(1)\left(\dfrac{dy}{dx}\right) + (y)(1))$. When we differentiate y^2, we get $2y\left(\dfrac{dy}{dx}\right)$ because we differentiate y . When we differentiate 13, we get 0. Now at this point we have to manipulate for $\dfrac{dy}{dx}$.

$$2x + x\left(\frac{dy}{dx}\right) + y + 2y\left(\frac{dy}{dx}\right) = 0$$

$$\begin{aligned}
x\left(\frac{dy}{dx}\right)+2y\left(\frac{dy}{dx}\right) &= -2x-y \\
(x+2y)\left(\frac{dy}{dx}\right) &= -2x-y \\
\frac{dy}{dx} &= \frac{-2x-y}{x+2y}.
\end{aligned}$$

Example 3.50 *If* $x^3-xy+y^2=10$, *find* $\dfrac{dy}{dx}$.

$$\begin{gathered}
x^3-xy+y^2=10 \\
3x^2-((x)(1)\left(\frac{dy}{dx}\right)+(y)(1))+2y\left(\frac{dy}{dx}\right)=0 \\
3x^2-(x\left(\frac{dy}{dx}\right)+y)+2y\left(\frac{dy}{dx}\right)=0 \\
3x^2-x\left(\frac{dy}{dx}\right)-y+2y\left(\frac{dy}{dx}\right)=0 \\
-x\left(\frac{dy}{dx}\right)+2y\left(\frac{dy}{dx}\right)=y-3x^2 \\
(-x+2y)\left(\frac{dy}{dx}\right)=y-3x^2 \\
\frac{dy}{dx}=\frac{y-3x^2}{-x+2y}.
\end{gathered}$$

Example 3.51 *If* $x^4-x^2y^3+y^3=15$, *find* $\dfrac{dy}{dx}$.

$$\begin{gathered}
x^4-x^2y^3+y^3=15 \\
4x^3-((x^2)(3y^2)\left(\frac{dy}{dx}\right)+(y^3)(2x))+3y^2\left(\frac{dy}{dx}\right)=0 \\
4x^3-(3x^2y^2\left(\frac{dy}{dx}\right)+2xy^3)+3y^2\left(\frac{dy}{dx}\right)=0 \\
4x^3-3x^2y^2\left(\frac{dy}{dx}\right)-2xy^3+3y^2\left(\frac{dy}{dx}\right)=0 \\
-3x^2y^2\left(\frac{dy}{dx}\right)+3y^2\left(\frac{dy}{dx}\right)=2xy^3-4x^3 \\
(-3x^2y^2+3y^2)\left(\frac{dy}{dx}\right)=2xy^3-4x^3 \\
\frac{dy}{dx}=\frac{2xy^3-4x^3}{-3x^2y^2+3y^2}.
\end{gathered}$$

Exercises 1.6

Q1 Find $\dfrac{dy}{dx}$ of the following functions:

(i) $x^3 + xy + y^2 = 11$.
(ii) $x^3 - xy + y^3 = 100$.
(iii) $x^3 - x^3y^3 + y^3 = 4$.

3.7 Parametric Differentiation

It is often necessary to find the rate of change of a function defined parametrically; that is, we want to calculate $\dfrac{dy}{dx}$ when both y and x are functions of the parameter t, i.e. $y = y(x)$ and $x = x(t)$, respectively. Provided that $\dfrac{dx}{dt} \neq 0$, here's the Parametric Differentiation rule:

$$\boxed{\text{If } y = y(x) \text{ and } x = x(t), \text{ then } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.}$$

Example 3.52 *If $x = t^3 + t$ and $y = 2 - t^2$, find $\dfrac{dy}{dx}$.*

In this example both y and x are in terms of the parameter t. We have to calculate $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$. $\dfrac{dx}{dt} = 3t^2 + 1$ and $\dfrac{dy}{dt} = -2t$ Therefore

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\
\frac{dy}{dx} &= \frac{-2t}{3t^2 + 1} \\
\frac{dy}{dx} &= -\frac{2t}{3t^2 + 1}.
\end{aligned}$$

Example 3.53 *If $x = \cos t$ and $y = -\sin t$, find $\dfrac{dy}{dx}$.*

Here both y and x are in terms of the parameter t. Thus we have to calculate $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ that is $\dfrac{dy}{dt} = -\sin t$ and $\dfrac{dx}{dt} = -\cos t$. Therefore

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\
\frac{dy}{dx} &= \frac{-\sin t}{-\cos t} \\
\frac{dy}{dx} &= \frac{\sin t}{\cos t}. \\
\frac{dy}{dx} &= \tan t.
\end{aligned}$$

Exercises 1.7

Q1 Find $\frac{dy}{dx}$ for the following curves:

(i) $x = t^3$ and $y = t^2 - t$.
(ii) $x = 3\cos t$ and $y = 3\sin t$.
(iii) $x = t + \sqrt{t}$ and $y = t - \sqrt{t}$.

3.8 Logarithmic Differentiation

Sometimes, differentiating a function following the product rule or the quotient rule could result in a fairly messy expression for the derivative. We can simplify things by taking logarithms of both sides before differentiating the equation. Given that $y = f(x)$ we take the natural logarithm on both sides, remembering to take absolute values, i.e. $\ln|y| = \ln|f(x)|$. Then we differentiate both sides using Implicit Differentiation. Here's the Logarithmic Differentiation rule:

$$\boxed{\text{If } y = f(x), \text{ then } \frac{1}{y} \cdot \frac{dy}{dx} = \frac{f'(x)}{f(x)}.}$$

Example 3.54 *If $y = x^3(1+x)^2$, find $\frac{dy}{dx}$ using Logarithmic Differentiation.*

In this example we have to use Logarithmic Differentiation. The first step is to take the logarithms of both sides, i.e. $\ln|y| = \ln|x^3(1+x)^2|$. The next step is to simplify the expression on the right hand side using the logarithmic identities. In this example

$$\begin{aligned}
\ln|y| &= \ln|x^3(1+x)^2| \\
\ln|y| &= \ln|x^3| + \ln|(1+x)^2| \\
\ln|y| &= 3\ln|x| + 2\ln|(1+x)|.
\end{aligned}$$

Now we differentiate both sides, using Implicit Differentiation.

$$\begin{aligned}
\frac{1}{y} \cdot \frac{dy}{dx} &= \frac{3}{x} + \frac{2}{x+1} \\
\frac{dy}{dx} &= \frac{3y}{x} + \frac{2y}{x+1}.
\end{aligned}$$

We want $\frac{dy}{dx}$ in terms of x. Thus we replace y with $x^3(1+x)^2$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{3(x^3(1+x)^2)}{x} + \frac{2(x^3(1+x)^2)}{x+1} \\
\frac{dy}{dx} &= 3(x^2(1+x)^2) + 2(x^3(1+x)).
\end{aligned}$$

Example 3.55 *If $y = x^x$, $x > 0$, find $\frac{dy}{dx}$ using Logarithmic Differentiation.*

In this example we have to use Logarithmic Differentiation. The first step is to take the logarithms of both sides, i.e. $\ln|y| = \ln|x^x|$. The next step is to simplify the expression on the right hand side using the logarithmic identities. In this example

$$\begin{aligned}
\ln|y| &= \ln(x^x) \\
\ln|y| &= x \ln x.
\end{aligned}$$

Now we differentiate both sides, using Implicit Differentiation.

$$\begin{aligned}
\frac{1}{y} \cdot \frac{dy}{dx} &= x \cdot \frac{1}{x} + (1) \ln x \\
\frac{dy}{dx} &= x \cdot \frac{y}{x} + y \ln x \\
\frac{dy}{dx} &= y + y \ln x \\
\frac{dy}{dx} &= y\,(1 + \ln x).
\end{aligned}$$

We want $\frac{dy}{dx}$ in terms of x. Thus we replace y with x^x.

$$\frac{dy}{dx} = x^x\,(1 + \ln x).$$

Exercises 1.8

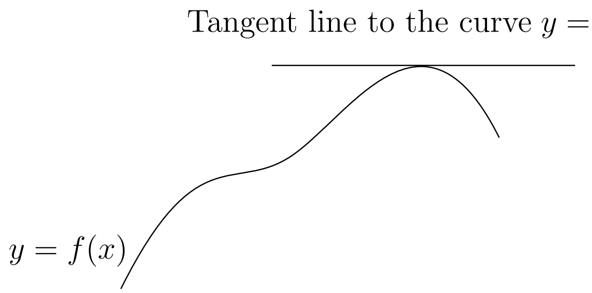
Q1 Find $\frac{dy}{dx}$ of the following functions, using Logarithmic Differentiation:

(i) $y = (x-1)^2 x^4$.
(ii) $y = x^3(2x+1)^2$.
(iii) $y = (x)^{e^x}$.

3.9 Equation Of A Tangent Line

3.9.1 Slope Of A Tangent Line

In general $\frac{dy}{dx}$ is a formula for calculating the slope of a tangent line to a curve at any point. Thus to find the slope of a tangent to a curve at a specific point, we simply sub in for x (and y) in $\frac{dy}{dx}$.



Example 3.56 *If* $y = \frac{(x^2+7x+3)}{(x^2+3)}$, *find the slope of the tangent to the curve at the point* $(0,0)$.

Using the quotient rule,

$$\frac{dy}{dx} = \frac{-7x^2+21}{(x^2+3)^2}.$$

Now to find the slope of the tangent to the curve, we simply sub in the x-coordinate of $(0,0)$ (which is 0) into $\frac{dy}{dx}$. Let's call m the slope of the tangent to the curve. Then

$$\begin{aligned} m &= \frac{dy}{dx} \text{ at } x = 0 \\ &= \frac{-7x^2+21}{(x^2+3)^2} \text{ at } x = 0 \\ &= \frac{-7(0)^2+21}{((0)^2+3)^2} \\ &= \frac{+21}{9} \\ &= \frac{7}{3}. \end{aligned}$$

Example 3.57 *If $x^2 + xy + y^2 = 3$, find the slope of the tangent to the curve at the point $(1, 1)$.*

Using Implicit Differentiation,

$$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}.$$

Now to find the slope of the tangent to the curve, we simply substitute the x-coordinate and the y-coordinate of $(1, 1)$ (which is 1 and 1, respectively) into $\frac{dy}{dx}$. Let's call m the slope of the tangent to the curve. Then

$$\begin{aligned} m &= \frac{dy}{dx} \text{ at } x = 1, y = 1 \\ &= \frac{-2x - y}{x + 2y} \text{ at } x = 1, y = 1 \\ &= \frac{-2(1) - 1}{1 + 2(1)} \\ &= \frac{-3}{3} \\ &= -1. \end{aligned}$$

Exercises 1.9.1

Q1 Find the slope of the tangent lines of the following functions at the corresponding points:

(i) $x^3 + xy - y^2 = 1$ at $(1, 1)$.
(ii) $y = \frac{x + 3}{x + 1}$ at $(1, 2)$.
(iii) $y = \frac{x^2 - 3x + 1}{x^2 + 1}$ at $(0, 1)$.

3.9.2 Equation Of A Tangent Line

In general the equation of a line who's slope is m and that passes through a point (x_1, y_1) is :

$$(y - y_1) = m(x - x_1).$$

Example 3.58 *If $y = \frac{(x^2 + 7x + 3)}{(x^2 + 3)}$, find the equation of the tangent to the curve at the point $(0, 0)$.*

We know from the previous section that the slope of the tangent line is $\frac{7}{3}$. i.e. $m = \frac{7}{3}$. Let x_1 be the x-coordinate of $(0, 0)$ and let y_1 be the y-coordinate of $(0, 0)$ and we get the equation of the tangent line is:

$$\begin{aligned}
(y - y_1) &= m(x - x_1) \\
(y - 0) &= \frac{7}{3}(x - 0) \\
y &= \frac{7}{3}x \\
3y &= 7x \\
7x - 3y &= 0.
\end{aligned}$$

Example 3.59 *If $x^2 + xy + y^2 = 3$, find the equation of the tangent to the curve at the point* $(1, 1)$.

We know from the previous section that the slope of the tangent line is -1. i.e. $m = -1$. Let x_1 be the x-coordinate of $(1, 1)$ and let y_1 be the y-coordinate of $(1, 1)$ and we get the equation of the tangent line is:

$$\begin{aligned}
(y - y_1) &= m(x - x_1) \\
(y - 1) &= -1(x - 1) \\
y - 1 &= -x + 1 \\
x + y - 2 &= 0.
\end{aligned}$$

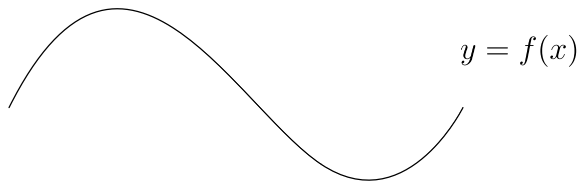
Exercises 1.9.2

Q1 Find the equation of the tangent lines of the following functions at the corresponding points:

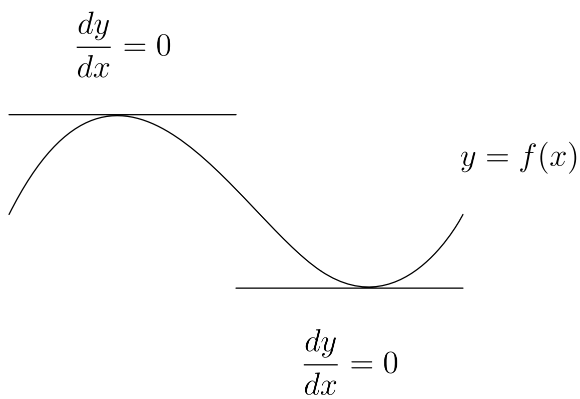
(i) $x^3 + xy - y^2 = 1$ at $(1, 1)$.
(ii) $y = \dfrac{x+3}{x+1}$ at $(1, 2)$.
(iii) $y = \dfrac{x^2 - 3x + 1}{x^2 + 1}$ at $(0, 1)$.

3.10 Maximum and Minimum Points of a Curve

Consider the function



The maximum and minimum points of the curve, if any, are where the slope of the tangent of the curve is 0.



Thus to determine the maximum and minimum points of a function :

$$\text{Let } \frac{dy}{dx} = 0 \text{ and solve for } x.$$

To determine whether a point (for which $\frac{dy}{dx} = 0$) is a maximum or a minimum, substitute the corresponding values of x into $\frac{d^2y}{dx^2}$ and if

- $\frac{d^2y}{dx^2} > 0$ at the point, then the point is a minimum.
- $\frac{d^2y}{dx^2} < 0$ at the point, then the point is a maximum.

Also to determine the points of inflection, we simply let $\frac{d^2y}{dx^2} = 0$ and solve for x.

Example 3.60 *Find the minimum point of the quadratic* $y = x^2 - 2x - 3$.

First find $\frac{dy}{dx}$ and let $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = 2x - 2$ and when we let $\frac{dy}{dx} = 0$, we get $2x - 2 = 0$, that is $x = 1$.

We know by the shape of this quadratic that this is a minimum point (i.e. U-shaped curve). However we can confirm this by finding $\frac{d^2y}{dx^2}$ (which is just the derivative of the first derivative). $\frac{d^2y}{dx^2} = 2$ which is greater than 0. Therefore it's a minimum point. Now we need to find a corresponding y-coordinate for $x = 1$. We do this by subbing $x = 1$ back into our original y. $y = f(1) = (1)^2 - 2(1) - 3 = 1 - 2 - 3 = -4$.

Therefore the quadratic $y = x^2 - 2x - 3$ has a minimum point at $(1, -4)$.

Example 3.61 *Find the max/min points and the points of inflection of the function* $y = x^3 - 6x^2 + 9x + 1$. *Hence roughly sketch the curve.*

First find $\frac{dy}{dx}$ and let $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = 3x^2 - 12x + 9 = 0$ and when we let $\frac{dy}{dx} = 0$, we get

$$3x^2 - 12x + 9 = 0$$
$$x^2 - 4x + 3 = 0$$
$$(x - 1)(x - 3) = 0$$
$$x = 1 \text{ or } x = 3.$$

Then find $\frac{d^2y}{dx^2}$ and determine its sign at the points for which $\frac{dy}{dx} = 0$. We have $\frac{d^2y}{dx^2} = 6x - 12$.

$$\frac{d^2y}{dx^2} \text{ (at } x = 1) = 6(1) - 12 = -6 < 0$$
$$\therefore \text{ The curve has a max point at } x = 1.$$

$$\frac{d^2y}{dx^2} \text{ (at } x = 3) = 6(3) - 12 = 6 > 0$$
$$\therefore \text{ The curve has a min point at } x = 3.$$

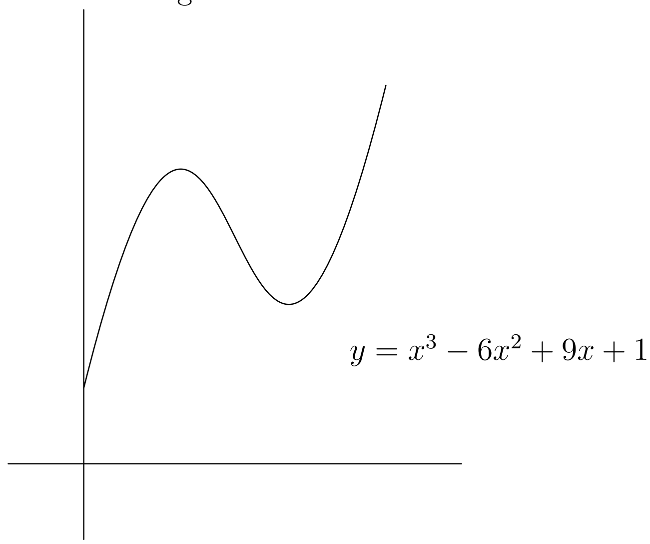
Also let $\frac{d^2y}{dx^2} = 0$, then $x = 2$. Therefore y has a point of inflection at $x = 2$.

Now we have to find corresponding y-coordinates to the x-coordinates of $x = 1$, $x = 2$ and $x = 3$.

$y = f(1) = (1)^3 - 6(1)^2 + 9(1) + 1 = 1 - 6 + 9 + 1 = 5$. Therefore $y = x^3 - 6x^2 + 9x + 1$ has a maximum point at $(1, 5)$.

$y = f(2) = (2)^3 - 6(2)^2 + 9(2) + 1 = 8 - 24 + 18 + 1 = 3$. Therefore $y = x^3 - 6x^2 + 9x + 1$ has a point of inflection at $(2, 3)$.

$y = f(3) = (3)^3 - 6(3)^2 + 9(3) + 1 = 27 - 54 + 27 + 1 = 1$. Therefore $y = x^3 - 6x^2 + 9x + 1$ has a minimum point at $(3, 1)$. Here's a rough sketch of the function:



Exercises 1.10

Q1 Find the maximum point of the quadratic $y = 4 - x^2$.

Q2 Find the max/min points and points of the following functions (also roughly sketch the functions):
(i) $y = x^3 - 3x^2 - 9x + 9$.
(ii) $y = x^3 - 6x^2 + 9x + 1$.

3.11 Answers

Exercises 1.1.1

Q1 (i) 5, (ii) 8, (iii) $\frac{1}{4}$, (iv) $\frac{1}{6}$, (v) 2, (vi) $\frac{3}{2}$.

Exercises 1.1.2

Q1 (i) 7, (ii) 5, (iii) $\frac{3}{4}$, (iv) 12.

Exercises 1.1.3

Q1 (i) 2 (ii) $\frac{1}{5}$, (iii) $+\infty$, (iv) $+\infty$, (v) $\frac{1}{2}$, (vi) 0.

Exercises 1.2

Q1 (i) $y' = 1$, (ii) $y' = 4$, (iii) $y' = 1$, (iv) $y' = 3$, (v) $y' = 4x$, (vi) $y' = 4x - 1$.

Exercises 1.5

Q1 (i) $y' = 7x^6 - 5 - \frac{4}{x^5}$, (ii) $y' = 5x^4 - 3\sin x + 4\cos x - 5e^x$, (iii) $y' = 8x^7 + \frac{7}{x}$,
(iv) $y' = -\frac{12}{x^4} + \frac{4}{x^5} - \frac{15}{x^6}$, (v) $y' = -\frac{1}{2x^{3/2}} - \frac{2}{3x^{4/3}} + \frac{1}{4x^{5/4}}$.

Exercises 1.5.1

Q1 (i) $y' = 49x^6 - 42x^5 + 28x^3 + 9x^2 - 6x$, (ii) $y' = 1 + \frac{3}{x^2}$, (iii) $y' = \sin x\,(4x^3 + 3x^2) + \cos x\,(x^4 + x^3)$,
(iv) $y' = x^2 e^x(3 + x)$, (v) $y' = e^x\left(\frac{1}{x} + 1\right)$, (vi) $y' = e^x(\sin x + \cos x)$, (vii) $y' = \cos x \ln x + \frac{\sin x}{x}$,
$(viii)$ $y' = \left(2x + 1 - \frac{1}{x^2}\right)\ln x + x + 1 + \frac{1}{x^2}$.

Exercises 1.5.2

Q1 (i) $y' = -\frac{2}{(x+1)^2}$, (ii) $y' = \frac{3(x^2-1)}{(x^2+1)^2}$, (iii) $y' = -\frac{\sin x + \cos x}{e^x}$, (iv) $y' = \frac{\cos x + x\sin x \ln x}{x\cos^2 x}$.

Exercises 1.5.3

Q1 (i) $y' = 3(x^7 + 7x^3)^2\,(7x^6 + 21x^2)$, (ii) $y' = 10(x^3 + x^2 + x + 7)^9\,(3x^2 + 2x + 1)$,
(iii) $y' = -\frac{8x+4}{(x^2+x+2)^5}$, (iv) $y' = 4\cos 4x$, (v) $y' = -(3x^2 + 2x + 1)\,\sin(x^3 + x^2 + x)$,
(vi) $y' = \frac{2x+2}{x^2+2x}$, (vii) $y' = (20x^4 + 1)\,e^{4x^5+x+1}$,
$(viii)$ $y' = -3(4x + 1)\sin(2x^2 + x + 7)\cos^2(2x^2 + x + 7)$.

Exercises 1.7

Q1 (i) $\dfrac{dy}{dx} = \dfrac{2t-1}{3t^2}$, (ii) $\dfrac{dy}{dx} = -\cot t$, (iii) $\dfrac{dy}{dx} = \dfrac{2\sqrt{t}-1}{2\sqrt{t}+1}$.

Exercises 1.8

Q1 (i) $\dfrac{dy}{dx} = 2x^4(x-1) + 4x^3(x-1)^2$, (ii) $\dfrac{dy}{dx} = 4x^3(2x+1) + 3x^2(2x+1)$,
(iii) $\dfrac{dy}{dx} = x^{(e^x-1)}e^x(1 + x\ln x)$.

Exercises 1.9.1

Q1 (i) $m = 4$, (ii) $m = -\dfrac{1}{2}$, (iii) $m = -3$.

Exercises 1.9.2

Q1 (i) $y = 4x - 3$, (ii) $2y + x = 5$, (iii) $y + 3x = 1$.

Exercises 1.10

Q1 $(0, 4)$ is a maximum point.

Q2 (i) $(3, -18)$, is a min point, $(-1, 14)$ is a max point, $(1, -2)$ is an inflexion point,
(ii) $(3, -26)$, is a min point, $(-1, 6)$ is a max point, $(1, -2)$ is an inflexion point.