University of Chester

# Maths Notes for first year Engineers

# 2. Eigenvalues and Eigenvectors

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# Chapter 2

# Eigenvalues & Eigenvectors

## 2.1 Eigenvalues

**Definition 2.1** *Let $A$ be a $n \times n$. A non-zero vector $\mathbf{v} \in \mathbb{R}^n$ is called an eigenvector of $A$ if and only if there exists a number $\lambda \in \mathbb{R}$ such that*

$$A\mathbf{v} = \lambda \mathbf{v}.$$

If such a number $\lambda$ exists, it is called an eigenvalue of $A$. The vector $\mathbf{v}$ is called the corresponding eigenvector to the eigenvalue of $\lambda$.

**Example 2.2** *Let* $A = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}$, $\lambda = -3$ *and* $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. *Then*

- $A\vec{v} = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$.
- $\lambda\vec{v} = -3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$.

*Therefore $\lambda = -3$ is an eigenvalue of $A$ with a corresponding eigenvector of* $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Consider $A\mathbf{v} = \lambda\mathbf{v}$. Now

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ A\mathbf{v} - \lambda\mathbf{v} &= \mathbf{0} \\ (A - \lambda I)\mathbf{v} &= \mathbf{0}. \end{aligned}$$

The above equation has nontrivial solutions if and only if $|A - \lambda I| = 0$. Thus to calculate the eigenvalues of the matrix $A$ we let $|A - \lambda I| = 0$ and solve for $\lambda$. Let $P(\lambda) = |A - \lambda I|$. Then $P(\lambda)$ is called the **characteristic polynomial** of $A$.

**Example 2.3** *Let* $A = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}$. *Calculate the eigenvalues of the matrix $A$.*

**Solution.** First we form the matrix $(A - \lambda I)$, then we solve $|A - \lambda I| = 0$.

- $(A - \lambda I) = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} (2-\lambda) & 5 \\ -1 & (-4-\lambda) \end{pmatrix}$.
-
$$\begin{aligned}
|A - \lambda I| &= 0 \\
\begin{vmatrix} (2-\lambda) & 5 \\ -1 & (-4-\lambda) \end{vmatrix} &= 0 \\
(2-\lambda)(-4-\lambda) - (5)(-1) &= 0 \\
-8 - 2\lambda + 4\lambda + \lambda^2 + 5 &= 0 \\
\lambda^2 + 2\lambda - 3 &= 0 \\
(\lambda + 3)(\lambda - 1) &= 0 \\
\lambda_1 = -3 \quad \lambda_2 &= 1.
\end{aligned}$$

Note that in this example the characteristic polynomial of $A$ is $P(\lambda) = \lambda^2 + 2\lambda - 3$. ♣

**Example 2.4** *Let* $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & -1 & -3 \end{pmatrix}$. *Calculate the eigenvalues of the matrix $A$.*

**Solution.** First we form the matrix $(A - \lambda I)$, then we solve $|A - \lambda I| = 0$.

- $(A - \lambda I) = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & -1 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (1-\lambda) & 2 & -1 \\ 2 & (3-\lambda) & 2 \\ -1 & -1 & (-3-\lambda) \end{pmatrix}$.
-
$$\begin{aligned}
|A - \lambda I| &= 0 \\
(1-\lambda)\begin{vmatrix} (3-\lambda) & 2 \\ -1 & (-3-\lambda) \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ -1 & (-3-\lambda) \end{vmatrix} - 1\begin{vmatrix} 2 & (3-\lambda) \\ -1 & -1 \end{vmatrix} &= 0 \\
(1-\lambda)[(3-\lambda)(-3-\lambda) + 2] - 2[2(-3-\lambda) + 2] - 1[-2 + (3-\lambda)] &= 0 \\
(1-\lambda)[\lambda^2 - 7] - 2[-4 - 2\lambda] - [1 - \lambda] &= 0 \\
-\lambda^3 + \lambda^2 + 12\lambda &= 0 \\
-\lambda(\lambda^2 - \lambda - 12) &= 0 \\
-\lambda(\lambda + 3)(\lambda - 4) &= 0 \\
\lambda_1 = 0 \quad \lambda_2 = -3 \quad \lambda_3 &= 4.
\end{aligned}$$

♣

**Exercises 2.1**

**Q1** Let $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. Calculate the eigenvalues of the matrices $A$, $B$ and $C$.

**Q2** Let $D = \begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}$ and $E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{pmatrix}$. Calculate the eigenvalues of the matrices $D$ and $E$.

## 2.2 Eigenvectors

Let $A$ be an $n \times n$ matrix with an eigenvalue of $\lambda$ and $\mathbf{v} \in \mathbb{R}^n$ is the corresponding eigenvector to the eigenvalue of $\lambda$. Recall the following equation holds:

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Thus we calculate the corresponding eigenvector to an eigenvalue of $\lambda$ by solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v} \in \mathbb{R}^n$. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$ and $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is equivalent to solving:

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \Longleftrightarrow \left( \begin{array}{ccccc|c} (a_{11} - \lambda) & a_{12} & a_{13} & \dots & a_{1n} & 0 \\ a_{21} & (a_{22} - \lambda) & a_{23} & \dots & a_{2n} & 0 \\ a_{31} & a_{32} & (a_{33} - \lambda) & \dots & a_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ a_{n1} & a_{n2} & a_{n3} & \dots & (a_{nn} - \lambda) & 0 \end{array} \right).$$

**Example 2.5** *Let* $A = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}$. *Calculate the eigenvectors corresponding to the eigenvalues of* $\lambda_1 = -3$ *and* $\lambda_2 = 1$.

**Solution.** Now $(A - \lambda I)\mathbf{v} = \mathbf{0} \Longrightarrow \left( \begin{array}{cc|c} (2 - \lambda) & 5 & 0 \\ -1 & (-4 - \lambda) & 0 \end{array} \right)$.

- $\lambda_1 = -3$

$$\left( \begin{array}{cc|c} 5 & 5 & 0 \\ -1 & -1 & 0 \end{array} \right) \sim \left( \begin{array}{cc|c} 5 & 5 & 0 \end{array} \right) \sim \left( \begin{array}{cc|c} 1 & 1 & 0 \end{array} \right)$$
$$\sim x + y = 0$$
$$\sim y = -x.$$

The eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ must satisfy $x + y = 0$. There are infinite sets of values that satisfy this equation. So we choose suitable values for $x, y$. For example, let $x = 1$, then $y = -1$. Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore the eigenvector corresponding to the eigenvalue of $\lambda_1 = -3$ is $\mathbf{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

- $\lambda_2 = 1$

$$\left( \begin{array}{cc|c} 1 & 5 & 0 \\ -1 & -5 & 0 \end{array} \right) \sim \left( \begin{array}{cc|c} 1 & 5 & 0 \end{array} \right)$$
$$\sim x + 5y = 0$$
$$\sim x = -5y.$$

Now we choose suitable values for $x, y$, satisfying $x = -5y$. Let $y = 1$, then $x = -5$. Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}.$$

Therefore the eigenvector corresponding to the eigenvalue of $\lambda_2 = 1$ is $\mathbf{v_2} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$.

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**Example 2.6** *Let* $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & -1 & -3 \end{pmatrix}$. *Calculate the eigenvectors corresponding to the eigenvalues of* $\lambda_1 = 0$, $\lambda_2 = -3$ *and* $\lambda_3 = 4$.

**Solution.** Now $(A - \lambda I)\mathbf{v} = \mathbf{0} \Longrightarrow \left( \begin{array}{ccc|c} (1-\lambda) & 2 & -1 & 0 \\ 2 & (3-\lambda) & 2 & 0 \\ -1 & -1 & (-3-\lambda) & 0 \end{array} \right)$.

For the purpose of this text, we assume that we have distinct eigenvalues. Hence we perform row operations until we reach the form

$$\left( \begin{array}{ccc|c} 1 & 0 & \bullet & 0 \\ 0 & 1 & \bullet & 0 \end{array} \right).$$

- $\lambda_1 = 0$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 3 & 2 & 0 \\ -1 & -1 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 3 & 2 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right) \quad r_3 + r_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_3 - r_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 3 & 2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right) \quad r_2 - r_1$$

$$\sim \left(\begin{array}{ccc|c} 0 & 1 & -4 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right) \quad r_1 - r_2$$

$$\sim \left(\begin{array}{ccc|c} 0 & 1 & -4 & 0 \\ 1 & 0 & 7 & 0 \end{array}\right) \quad r_2 - r_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right) \quad r_1 \leftrightarrow r_2$$

$$\sim \begin{array}{c} x + 7z = 0 \\ y - 4z = 0 \end{array}$$

$$\sim \begin{array}{c} x = -7z \\ y = 4z \end{array}$$

The eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ must satisfy $x = -7z$ and $y = 4z$ There are infinite sets of values that satisfy these equations. So we choose suitable values for $x, y, z$. For example, let $z = 1$, then $y = 4$ and $x = -7$. Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 4 \\ 1 \end{pmatrix}.$$

Therefore the eigenvector corresponding to the eigenvalue of $\lambda_1 = 0$ is $\mathbf{v_1} = \begin{pmatrix} -7 \\ 4 \\ 1 \end{pmatrix}$.

- $\lambda_2 = -3$

$$
\begin{aligned}
\left(\begin{array}{ccc|c} 4 & 2 & -1 & 0 \\ 2 & 6 & 2 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right) &\sim \left(\begin{array}{ccc|c} 10 & 10 & 0 & 0 \\ 2 & 6 & 2 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right) \quad 2r_1 + r_2 \\
&\sim \left(\begin{array}{ccc|c} 2 & 6 & 2 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right) \quad \tfrac{1}{2}r_1 \\
&\sim \left(\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right) \quad r_1 + r_2 \\
&\sim \left(\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \quad r_2 \times (-1) \\
&\sim \begin{array}{c} 2y + z = 0 \\ x + y = 0 \end{array} \\
&\sim \begin{array}{c} z = -2y \\ x = -y \end{array}
\end{aligned}
$$

As explained above, we choose suitable values for $x, y, z$, satisfying $z = -2y$ and $x = -y$. Let $y = 1$, then $z = -2$ and $x = -1$. Thus

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}.
$$

Therefore the eigenvector corresponding to the eigenvalue of $\lambda_2 = -3$ is $\mathbf{v_2} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$.

- $\lambda_3 = 4$

$$\left(\begin{array}{ccc|c} -3 & 2 & -1 & 0 \\ 2 & -1 & 2 & 0 \\ -1 & -1 & -7 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 9 & -6 & 3 & 0 \\ -10 & 5 & -10 & 0 \\ -1 & -1 & -7 & 0 \end{array}\right) \begin{array}{l} r_1 \times (-3) \\ r_2 \times (-5) \\ \phantom{0} \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 9 & -6 & 3 & 0 \\ -1 & -1 & -7 & 0 \\ -1 & -1 & -7 & 0 \end{array}\right) \quad r_2 + r_1$$

$$\sim \left(\begin{array}{ccc|c} 9 & -6 & 3 & 0 \\ -1 & -1 & -7 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 3 & -2 & 1 & 0 \\ -1 & -1 & -7 & 0 \end{array}\right) \quad r_1/3$$

$$\sim \left(\begin{array}{ccc|c} 5 & 0 & 15 & 0 \\ -1 & -1 & -7 & 0 \end{array}\right) \quad r_1 - 2r_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ -1 & -1 & -7 & 0 \end{array}\right) \quad r_1/5$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right) \quad r_2 + r_1$$

$$\sim \begin{array}{l} x + 3z = 0 \\ y - 4z = 0 \end{array}$$

$$\sim \begin{array}{l} x = -3z \\ y = -4z \end{array}$$

Again we choose suitable values for $x, y, z$, satisfying $x = -3z$ and $y = -4z$. Let $z = 1$, then $x = -3$ and $y = -4$. Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix}.$$

Therefore the eigenvector corresponding to the eigenvalue of $\lambda_3 = 4$ is $\mathbf{v_3} = \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix}$.

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**Exercises 2.2**

**Q1** Let $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. Calculate the eigenvectors corresponding to the eigenvalues computed in **Exercise 2.1** of the matrices $A$, $B$ and $C$.

**Q2** Let $D = \begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}$ and $E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{pmatrix}$. Calculate the eigenvectors corresponding to the eigenvalues computed in **Exercise 2.1** of the matrices $D$ and $E$.

## 2.3 Diagonalization

Let $A$ be a square matrix, then there exists an invertible matrix $E$ and a diagonal matrix $D$ such that

$$AE = ED$$

where $E$ is a matrix consisting of the eigenvectors of $A$ and the eigenvalues of $A$ are the diagonal entries of $D$ (i.e. $E = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ \vdots & \vdots & & \vdots \end{pmatrix}$ $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are the corresponding eigenvectors of the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $A$.
If we manipulate the above equation for $D$,

$$\begin{aligned} ED &= AE \\ E^{-1}ED &= E^{-1}AE \\ ID &= E^{-1}AE \\ D &= E^{-1}AE. \end{aligned}$$

**Example 2.7** *Let* $A = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}$, *diagonalize* $A$.

**Solution.** We know (from Example 2.3 and 2.5) that the two eigenvalues of $A$ are $\lambda_1 = -3$ and $\lambda_2 = 1$ and the 2 corresponding eigenvectors are $\mathbf{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{v_2} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$. Thus $E = \begin{pmatrix} 1 & -5 \\ -1 & 1 \end{pmatrix}$ and

$$\begin{aligned} D = E^{-1}AE &= \frac{1}{|E|}E^*AE \\ &= \begin{pmatrix} 1 & -5 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} 1 & -5 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{(1)(1)-(-5)(-1)} \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} 1 & -5 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{-4} \begin{pmatrix} -3 & -15 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -5 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{-4} \begin{pmatrix} 12 & 0 \\ 0 & -4 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \end{aligned}$$

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**Example 2.8** *Let* $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & -1 & -3 \end{pmatrix}$, *diagonalize* $A$.

**Solution.** We know (from Example 2.4 and 2.6) that the three eigenvalues of $A$ are $\lambda_1 = 0$, $\lambda_2 = -3$ and $\lambda_3 = 4$. Also the three corresponding eigenvectors are $\mathbf{v_1} = \begin{pmatrix} -7 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$ and $\mathbf{v_3} = \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix}$. Thus $E = \begin{pmatrix} -7 & -1 & -3 \\ 4 & 1 & -4 \\ 1 & -2 & 1 \end{pmatrix}$ and

$$\begin{aligned}
D = E^{-1}AE &= \frac{1}{|E|}E^*AE \\
&= \frac{1}{84}\begin{pmatrix} -7 & 7 & 7 \\ -8 & -4 & -40 \\ -9 & -15 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & -1 & -3 \end{pmatrix}\begin{pmatrix} -7 & -1 & -3 \\ 4 & 1 & -4 \\ 1 & -2 & 1 \end{pmatrix} \\
&= \frac{1}{84}\begin{pmatrix} 0 & 0 & 0 \\ 24 & 12 & 120 \\ -36 & -60 & -12 \end{pmatrix}\begin{pmatrix} -7 & -1 & -3 \\ 4 & 1 & -4 \\ 1 & -2 & 1 \end{pmatrix} \\
&= \frac{1}{84}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -252 & 0 \\ 0 & 0 & 336 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.
\end{aligned}$$

♣

**Exercises 2.3**

**Q1** Let $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. Diagonalize the matrices $A$, $B$ and $C$.

**Q2** Let $D = \begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}$ and $E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{pmatrix}$. Diagonalize the matrices $D$ and $E$.

## 2.4 Answers

**Exercises 2.1**

**Q1** Matrix $A$: $\lambda_1 = -1$, $\lambda_2 = -2$, matrix $B$: $\lambda_1 = -3$, $\lambda_2 = 4$, matrix $C$: $\lambda_1 = 5$, $\lambda_2 = -1$.

**Q2** Matrix $D$: $\lambda_1 = 3$, $\lambda_2 = -5$, $\lambda_3 = 6$, matrix $E$: $\lambda_1 = 1$, $\lambda_2 = 2$ ,$\lambda_3 = 4$.

**Exercises 2.2**

**Q1** Matrix $A$: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, matrix $B$: $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, matrix $C$: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

**Q2** Matrix $D$: $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 6 \\ 16 \end{pmatrix}$, matrix $B$: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

**Exercises 2.3**

**Q1** $\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, $\begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}$, $\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$.

**Q2** $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.