University of Chester

Maths Notes for first year Engineers

10. Sequences and Series

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Contents

Chapter 10

Sequences and Series

10.1 Sequences

Definition 10.1 *An infinite* ***sequence*** *of numbers is a function whose domain is the set of integers greater than or equal to some integer* n_0*.*

Informally, a sequence is an ordered set.

Example 10.2 *Consider the sequence* $\{a_n = n\}$ *for* $n \geq 0$*.*

This sequence look like:

$$\{a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3, \ldots, a_n = n, \ldots\} \text{ or}$$
$$\{0, 1, 2, 3, \ldots, \underbrace{n}_{\text{This is called the } n\text{-th or the general term of the sequence}}, \ldots\}.$$

Example 10.3 *Consider the sequence* $\{a_n = n^2\}$ *for* $n \geq 0$*.*

This sequence look like:

$$\{0, 1, 4, 9, \ldots, n^2, \ldots\}.$$

Example 10.4 *Consider the sequence* $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$*. Find the general term of this sequence.*

$a_n = \frac{1}{n}$ for $n \geq 1$.

Example 10.5 *Consider the sequence* $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \ldots\}$*. Find the general term of this sequence.*

$a_n = \frac{n-1}{n}$ for $n \geq 1$.

Example 10.6 *Consider the sequence* $\{0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4} \ldots\}$*. Find the general term of this sequence.*

$a_n = (-1)^{n+1}\left(\frac{n-1}{n}\right)$ for $n \geq 1$.

10.1.1 Arithmetic Sequence

Definition 10.7 *An **Arithmetic** sequence is a sequence where each term after the first is obtained by adding a non-zero constant (called the common difference) to the previous one.*

Let $a_1 = a$ be the first term and d be the common difference. Then an arithmetic sequence looks like:

$$\{a, a+d, a+2d, a+3d, \ldots\}.$$

Then the general term of an arithmetic sequence is $a_n = a + (n-1)d$ for $n \geq 1$.

10.1.2 Geometric Sequence

Definition 10.8 *A **Geometric** sequence is a sequence where each term after the first is obtained by multiplying the previous one by a non-zero constant called the common ratio.*

Let $a_1 = a$ be the first term and r be the common ration. Then a geometric sequence looks like:

$$\{a, ar, ar^2, ar^3, \ldots\}.$$

Then the general term of a geometric sequence is $a_n = ar^{n-1}$ for $n \geq 1$.

Definition 10.9 *A **series** is the sum of a sequence of terms.*

Definition 10.10 *The **partial sums** of the series form a sequence*

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= \sum_{i=1}^{n} a_i \\
&\vdots
\end{aligned}$$

of real numbers, each defined as a finite sum.

Now we set out to formulate S_n of a geometric sequence.

$$\begin{aligned}
S_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\
\underline{-rS_n} &= \underline{\quad -ar - ar^2 - \cdots - ar^{n-1} - ar^n} \\
S_n - rS_n &= a - ar^n \\
S_n(1-r) &= a(1-r^n) \\
S_n &= \frac{a(1-r^n)}{(1-r)} \qquad r \neq 1.
\end{aligned}$$

Definition 10.11 *We say that the series* $\sum_{n=1}^{\infty} a_n$ *converges to* S *if*

$$\lim_{n\to\infty} (S_n) = S.$$

Otherwise, we say that the series diverges.

Theorem 10.12 *Consider* $\sum_{n=1}^{\infty} ar^{n-1}$. *Then* $\sum_{n=1}^{\infty} ar^{n-1}$

(*i*) *converges to* $\left(\frac{a}{1-r}\right)$ *if* $|r| < 1$.

(*ii*) *diverges if* $|r| \geq 1$.

Proof. If $\{a_n\}$ is a geometric sequence, then $S_n = \frac{a(1-r^n)}{(1-r)}$ and

$$\begin{aligned}
\lim_{n\to\infty} (S_n) &= \lim_{n\to\infty} \left(\frac{a(1-r^n)}{(1-r)}\right) \\
&= \lim_{n\to\infty} \left(\frac{a}{(1-r)} - \frac{ar^n}{(1-r)}\right) \\
&= \lim_{n\to\infty} \left(\frac{a}{1-r}\right) - \lim_{n\to\infty} \left(\frac{ar^n}{1-r}\right) \\
&= \frac{a}{1-r} - \left(\frac{a}{1-r}\right) \lim_{n\to\infty} (r^n).
\end{aligned}$$

(i) If $|r| < 1$, then $\lim_{n\to\infty} (r^n) = 0$ and $\lim_{n\to\infty} (S_n) = \frac{a}{1-r}$.

(ii) If $|r| \geq 1$, then $\lim_{n\to\infty} (r^n) = \pm\infty$ and $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

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10.1.3 Applications

Example 10.13 *Rewrite* $0.7777777\ldots$ *as a fraction.*

$$\begin{aligned}
0.7777777\ldots &= 0.7 + 0.07 + 0.007 + \cdots \\
&= 7\left(\frac{1}{10}\right) + 7\left(\frac{1}{100}\right) + 7\left(\frac{1}{1000}\right) + \cdots \\
0.7777777\ldots &= 7\left(\frac{1}{10}\right) + 7\left(\frac{1}{10}\right)^2 + 7\left(\frac{1}{10}\right)^3 + \cdots \\
&= \frac{7}{10}\left(1 + \left(\frac{1}{10}\right) + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots\right) \\
&= \sum_{n=1}^{\infty} \frac{7}{10}\left(\frac{1}{10}\right)^{n-1}.
\end{aligned}$$

We know that $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ if $|r| < 1$. Therefore $\sum_{n=1}^{\infty} \frac{7}{10}\left(\frac{1}{10}\right)^{n-1} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{9}$.

Example 10.14 *Rewrite* $0.232323\ldots$ *as a fraction.*

$$\begin{aligned}
0.232323\ldots &= 0.23 + 0.0023 + 0.000023 + \cdots \\
&= 23\left(\frac{1}{100}\right) + 23\left(\frac{1}{100}\right)^2 + 23\left(\frac{1}{10}\right)^3 + \cdots \\
&= \frac{23}{100}\left(1 + \left(\frac{1}{100}\right) + \left(\frac{1}{100}\right)^2 + \left(\frac{1}{100}\right)^3 + \cdots\right) \\
&= \sum_{n=1}^{\infty} \frac{23}{100}\left(\frac{1}{100}\right)^{n-1} \\
&= \frac{\frac{23}{100}}{1-\frac{1}{100}} \\
&= \frac{23}{99}.
\end{aligned}$$

10.1.4 Tests for Convergence and Divergence of $\sum_{n=0}^{\infty} a_n$

In this section, we discuss different tests to determine if $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

Theorem 10.15 *The p-series* $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$ $(p \in \mathbb{R})$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

Theorem 10.16 (*The Ratio Test*) *Let* $\sum_{n=0}^{\infty} a_n$ *be a series with positive terms and suppose that*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho.$$

Then

(a) *the series converges absolutely if* $\rho < 1$.

(b) *the series diverges if* $\rho > 1$ *or if* ρ *is infinite.*

(c) *the test is inconclusive if* $\rho = 1$.

Example 10.17 *Investigate the convergence of* $\sum_{n=1}^{\infty} \frac{2^n}{n3^n}$.

$a_n = \frac{2^n}{n3^n} \Longrightarrow a_{n+1} = \frac{2^{n+1}}{(n+1)3^{n+1}}$. Thus

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)3^{n+1}} . \frac{n3^n}{2^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{2n}{3(n+1)} \right| \\
&= \frac{2}{3} \lim_{n\to\infty} \left| \frac{n}{n+1} \right| \\
&= \frac{2}{3} |1| \\
&= \frac{2}{3} < 1.
\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} \frac{2^n}{n3^n}$ is convergent.

Example 10.18 *Investigate the convergence of* $\sum_{n=1}^{\infty} \frac{n!}{e^n}$.

$a_n = \dfrac{n!}{e^n} \Longrightarrow a_{n+1} = \dfrac{(n+1)!}{e^{n+1}} = \dfrac{(n+1)n!}{e^{n+1}}$. Thus

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left| \frac{(n+1)n!}{e^{n+1}} \cdot \frac{e^n}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{n+1}{e} \right| \\
&= \frac{1}{e} \lim_{n\to\infty} |n+1| \\
&= \infty
\end{aligned}$$

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^n}$ is divergent.

Theorem 10.19 (*The n-th Root Test*) *Let* $\displaystyle\sum_{n=0}^{\infty} a_n$ *be a series with positive terms and suppose that*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Then

(a) *the series converges if* $\rho < 1$.

(b) *the series diverges if* $\rho > 1$ *or rho is infinite.*

(c) *the test is inconclusive if* $\rho = 1$.

Example 10.20 *Investigate the convergence of* $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln(n))^n}{n^n}$.

$a_n = \dfrac{(\ln(n))^n}{n^n}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \left[\frac{(\ln(n))^n}{n^n} \right]^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left[\left(\frac{\ln(n)}{n} \right)^n \right]^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\ln(n)}{n} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} \qquad \text{by L'H\^ospital's rule} \\
&= \lim_{n\to\infty} \frac{1}{n} \\
&= 0 < 1.
\end{aligned}$$

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln(n))^n}{n^n}$ is convergent.

10.1.5 Exercises

Q1 Rewrite the following as fractions:

$(i)\ 0.\dot{9} \quad (ii)\ 0.\dot{2}\dot{1} \quad (iii)\ 0.\dot{2}\dot{3}\dot{4} \quad (iv)\ 1.\dot{4}\dot{1}\dot{4}.$

Q2 Use an appropriate test to each of the following series for convergence and divergence:

$$(i)\ \sum_{n=1}^{\infty}\frac{5}{n+1} \quad (ii)\ \sum_{n=2}^{\infty}\frac{\ln(n)}{n} \quad (iii)\ \sum_{n=1}^{\infty}\frac{e^n}{1+e^{2n}} \quad (iv)\ \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}(\sqrt{n}+1)}$$

$$(v)\ \sum_{n=1}^{\infty}\frac{n!n!}{(2n)!} \quad (vi)\ \sum_{n=1}^{\infty}\frac{3^n}{n^3 2^n} \quad (vii)\ \sum_{n=1}^{\infty}\frac{n^{\sqrt{2}}}{2^n} \quad (viii)\ \sum_{n=1}^{\infty}n^2e^{-n} \quad (ix)\ \sum_{n=1}^{\infty}n!e^{-n}$$

$$(x)\ \sum_{n=1}^{\infty}\frac{n!}{10^n} \quad (xi)\ \sum_{n=1}^{\infty}\frac{n^{10}}{10^n} \quad (xii)\ \sum_{n=1}^{\infty}\frac{n\ln(n)}{2^n} \quad (xiii)\ \sum_{n=1}^{\infty}n^3e^{-n} \quad (xiv)\ \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)^n$$

$$(xv)\ \sum_{n=1}^{\infty}\frac{n^n}{(2^n)^2} \quad (xvi)\ \sum_{n=1}^{\infty}\frac{n}{(\ln(n))^n}.$$

10.2 Series

10.2.1 Power Series

Definition 10.21 *An expression of the form*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-1) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

*is called a **power series** centered at $x = a$.*

Theorem 10.22 *If $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $a-\delta < x < a+\delta$ for some $\delta > 0$, it defines a function $f(x)$:*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad a-\delta < x < a+\delta.$$

Such a function has derivatives of all orders inside the interval of convergence.

Example 10.23 *Consider $f(x) = \dfrac{1}{1-x}$. Can we rewrite $f(x) = \dfrac{1}{1-x}$ as $\sum_{n=0}^{\infty} c_n(x-a)^n$, i.e. $\sum_{n=0}^{\infty} c_n(x-a)^n$ converge?*

Recall that an infinite geometric series $\sum_{n=1}^{\infty} ar^{n-1}$, this series converges to $\dfrac{a}{1-r}$. Therefore $\dfrac{a}{1-r} = \sum_{n=1}^{\infty} ar^{n-1}$. Therefore

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=1}^{\infty} x^{n-1} \qquad a = 1,\ r = x \\ &= 1 + x + x^2 + x^3 + x^4 + \cdots \\ &= \sum_{n=0}^{\infty} x^n.\end{aligned}$$

Clearly $\sum_{n=0}^{\infty} x^n$ converges for $|x| < 1 \iff -1 < x < 1$, since it is an infinite geometric series. Therefore

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + x^3 + x^4 + \cdots .\end{aligned}$$

If we assume that $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges. Then

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n(x-a)^n \\ &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots . \end{aligned}$$

How do we obtain $c_0, c_1, c_2, c_3, \ldots, c_n, \ldots$?
If $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$, then

$$\begin{aligned} f(a) &= c_0 + c_1(a-a) + c_2(a-a)^2 + c_3(a-a)^3 + \cdots + c_n(a-a)^n + \cdots \\ &= c_0 \\ &= 0!c_0. \end{aligned}$$

Then

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + \cdots +$$

and

$$\begin{aligned} f'(a) &= c_1 \\ &= 1!c_1. \end{aligned}$$

Also

$$f''(x) = 2c_2 + 6c_3(x-a)^1 + \cdots + n(n-1)c_n(x-a)^{n-2} + \cdots +$$

and

$$\begin{aligned} f''(a) &= 2c_2 \\ &= 2!c_2. \end{aligned}$$

Also

$$f'''(x) = 6c_3 + \cdots + n(n-1)(n-2)c_n(x-a)^{n-3} + \cdots +$$

and

$$\begin{aligned} f''(a) &= 6c_3 \\ &= 3!c_3. \end{aligned}$$

Therefore $f^n(a) = n!c_n$ ($f^{(n)}(a)$ is the n-th derivative of f at $x = a$) and $c_n = \dfrac{f^{(n)}(a)}{n!}$. Thus

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} c_n(x-a)^n \qquad \text{since } \sum_{n=0}^{\infty} c_n(x-a)^n \text{ converges}\\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n\\
&= \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \cdots\\
&= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots .
\end{aligned}$$

10.2.2 Taylor series

Definition 10.24 *Let $f(x)$ be a function that has derivatives of all orders on its domain-that is, the n-th derivative, $f^{(n)}(x)$, exists for $n = 1, 2, 3, \ldots$. We define the **Taylor** series expansion of $f(x)$ about the point a to be*

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n\\
&= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .
\end{aligned}$$

Example 10.25 (*i*) *Find the Taylor series expansion of $f(x) = \dfrac{1}{x}$ at $a = 3$.*

(*ii*) *Where does this series converge absolutely?*

Solution: (*i*)

- $f(x) = \dfrac{1}{x}$. Then $f(3) = \dfrac{1}{3}$.
- $f(x) = x^{-1}$ and $f'(x) = -x^{-2} = -\dfrac{1}{x^2}$. Thus $f'(3) = -\dfrac{1}{9}$
- $f'(x) = -x^{-2}$ and $f''(x) = 2x^{-3} = \dfrac{2}{x^3}$. Thus $f''(3) = \dfrac{2}{27}$
- $f''(x) = 2x^{-3}$ and $f''(x) = -6x^{-4} = -\dfrac{6}{x^4}$. Thus $f'''(3) = -\dfrac{6}{81}$.

The Taylor series expansion of $f(x) = \dfrac{1}{x}$ at $a = 3$ is:

$$\begin{aligned}
f(x) &= f(3) + \frac{f'(3)}{1!}(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + \cdots\\
&= \frac{1}{3} - \frac{1}{9}(x-3) + \frac{1}{27}(x-3)^2 - \frac{1}{81}(x-3)^3 + \cdots\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(x-3)^n}{3^{n+1}}.
\end{aligned}$$

(ii) Using the ratio test, the series converges absolutely if:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &< 1\\
\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-3)^{n+1}}{3^{n+2}}\frac{3^{n+1}}{(-1)^n(x-3)^n}\right| &< 1\\
\lim_{n\to\infty}\left|\frac{(x-3)}{3}\right| &< 1\\
\left|\frac{(x-3)}{3}\right| &< 1\\
\frac{|(x-3)|}{3} &< 1\\
|(x-3)| &< 3\\
-3 < (x-3) &< 3\\
-3+3 < x &< 3+3\\
0 < x &< 6.
\end{aligned}$$

Example 10.26 *Find the Taylor Series expansion of $f(x) = e^x$ about $x = a$.*

(*i*) *Find the Taylor series expansion of $f(x) = e^x$ at $x = a$.*

(*ii*) *Where does this series converge?*

Solution (i) $f(x) = e^x$. Clearly $f^{(n)}(x) = e^x$ and $f^{(n)}(a) = e^a$. Thus the Taylor series expansion of $f(x) = e^x$ about $x = a$ is:

$$\begin{aligned}
f(x) &= e^a + \frac{e^a(x-a)}{1!} + \frac{e^a(x-a)^2}{2!} + \cdots\\
&= \sum_{n=0}^{\infty}\frac{e^a(x-a)^n}{n!}.
\end{aligned}$$

(ii) Using the ratio test, the series converges absolutely if:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &< 1\\
\lim_{n\to\infty}\left|\frac{(n+1)!}{e^a(x-a)^{n+1}}\frac{e^a(x-a)^n}{n!}\right| &< 1\\
\lim_{n\to\infty}\left|\frac{(x-a)}{n+1}\right| &< 1\\
|(x-a)|\lim_{n\to\infty}\left|\frac{1}{n+1}\right| &< 1\\
|(x-a)|\cdot 0 &< 1\\
0 &< 1.
\end{aligned}$$

Therefore this series converges for all $x \in \mathbb{R}$.

10.2.3 Maclaurin series

Definition 10.27 *Let $f(x)$ be a function that has derivatives of all orders on its domain. We define the **Maclaurin** series expansion of $f(x)$ to be*

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}.$$

Note that, this is the Taylor series expansion when $a = 0$.

Example 10.28 *Find the Maclaurin series expansion of $f(x) = e^x$ and find an approximation for e.*

We can obtain the Maclaurin series expansion of $f(x) = e^x$ by replacing a with 0 in the previous example. Therefore

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To find an approximation for $e = e^1 = 2.718281828$, we simply replace x with 1 on both sides of the equation. However the more terms in the expansion, gives a better approximation.

$$\begin{aligned} e^x &= 1 + \frac{(1)}{1!} + \frac{(1)^2}{2!} + \frac{(1)^3}{3!} + \frac{(1)^4}{4!} + \frac{(1)^5}{5!} + \frac{(1)^6}{6!} \\ &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} \\ &= 2.718055556. \end{aligned}$$

You can see that this okay to 3 decimal places. **Question** How many more term would we have to add to get 2.718281828?

Example 10.29 *Find the Maclaurin series expansion of $f(x) = \sin(x)$.*

$f(x) = \sin(x),\ f(0) = 0.$
$f'(x) = \cos(x),\ f'(0) = 1.$
$f''(x) = -\sin(x),\ f''(0) = 0.$
$f'''(x) = -\cos(x),\ f'''(0) = -1.$
$f^4(x) = \sin(x),\ f^4(0) = 0.$

Clearly this pattern will be repeated again and again. Therefore

$$\begin{aligned} \sin(x) &= 0 + \frac{1 \cdot x}{1!} + \frac{0 \cdot x^2}{2!} + \frac{-1 \cdot x^3}{3!} + \frac{0 \cdot x^4}{4!} + \frac{1 \cdot x^5}{5!} + \frac{0 \cdot x^6}{6!} + \frac{-1 \cdot x^7}{7!} + \cdots \\ &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}. \end{aligned}$$

Example 10.30 *Find the Maclaurin series expansion of* $f(x) = \tan^{-1}(x)$ *and find an approximation for* π.

$f(x) = \tan^{-1}(x),\ f(0) = 0.$
$f'(x) = \dfrac{1}{1+x^2},\ f'(0) = 1.$

We know that $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Therefore

$$\begin{aligned} f'(x) = \frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \cdots \\ &= 1 - x^2 + x^4 - x^6 + x^8 - \cdots . \end{aligned}$$

Thus $f''(x) = -2x + 4x^3 - 6x^5 + 8x^7 - \cdots,\ f''(0) = 0.$
$f'''(x) = -2 + 12x^2 - 30x^4 + 56x^6 - \cdots,\ f'''(0) = -2.$
$f^4(x) = 24x - 120x^3 + 336x^5 - \cdots,\ f^4(0) = 0.$
$f^5(x) = 24 - 360x^2 + 1880x^4 - \cdots,\ f^5(0) = 24.$
$f^6(x) = -720x + 75200x^3 - \cdots,\ f^6(0) = 0.$
$f^7(x) = -720 + 22560x^2 - \cdots,\ f^7(0) = -720.$

Therefore

$$\begin{aligned} \tan^{-1}(x) &= 0 + \frac{(1)x}{1!} + \frac{(0)x^2}{2!} + \frac{(-2)x^3}{3!} + \frac{(0)x^4}{4!} + \frac{(24)x^5}{5!} + \frac{(0)x^6}{6!} + \frac{(-720)x^7}{7!} + \cdots \\ &= \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}. \end{aligned}$$

To find an approximation for π, we use the fact that $\tan^{-1}(1) = \dfrac{\pi}{4}$. Therefore

$$\begin{aligned} \tan^{-1}(1) &= \frac{1}{1} - \frac{1^3}{3} + \frac{1^5}{5} - \frac{1^7}{7} + \frac{1^9}{9} - \frac{1^{11}}{11} + \frac{1^{13}}{13} - \frac{1^{15}}{15} + \frac{1^{17}}{17} - \frac{1^{19}}{19} + \frac{1^{21}}{21} - \frac{1^{23}}{23} \\ \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \frac{1}{21} - \frac{1}{23} \end{aligned}$$

$$\begin{aligned} \frac{\pi}{4} &= 0.7646006915 \\ \pi &= 4(0.7646006915) = 3.058402766 \end{aligned}$$

Again, the more terms that we add, the closer we get to $\pi = 3.141592654$.

Example 10.31 *Find the Maclaurin series expansion of $f(x) = \cosh(x)$.*

$\cosh(x) = \left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{1}{2}\left(e^x + e^{-x}\right)$. We know that the Maclaurin series expansion of e^x is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. To find the Maclaurin series expansion of e^{-x}, we simply replace x with $-x$ in e^x.

$$\begin{aligned}
e^{-x} &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \\
&= 1 + \frac{(-x)}{1!} + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \cdots \\
&= 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.
\end{aligned}$$

Thus

$$\begin{aligned}
\cosh(x) &= \frac{1}{2}\left[e^x + e^{-x}\right] \\
&= \frac{1}{2}\left[\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right) + \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right)\right] \\
&= \frac{1}{2}\left[2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots\right] \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!}.
\end{aligned}$$

10.2.4 Exercises

Q1 Find the Taylor series expansion of:

(i) $f(x) = \dfrac{1}{x^2}$ at $a = 1$.

(ii) $f(x) = \dfrac{x}{1-x}$ at $a = 2$.

(iii) $f(x) = \dfrac{1}{x}$ at $a = 1$.

Q2

(i) Find the Taylor series expansion of e^x about $x = a$.

(ii) Hence find the Maclaurin series expansion of e^x.

(iii) Find an approximation to e^1.

Q3 Find the Maclaurin series expansion of $\tan^{-1}(x)$.

Q4

(i) Find the first 5-terms of the Maclaurin series expansion of $\sqrt{1+x}$.

(ii) Hence find an approximation to $\sqrt{2}$.

10.3 Answers

Exercises 10.1.6

Q1 (i) 1 (ii) $\frac{7}{33}$ (iii) $\frac{26}{111}$ (iv) $\frac{140}{99}$.

Q2 (i) diverges (ii) diverges (iii) converges (iv) diverges

(v) converges (vi) diverges (vii) converges $(viii)$ converges

(ix) diverges (x) diverges (xi) converges (xii) converges

Exercises 10.2.4

Q1

(i) $f(x) = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n.$

(ii) $f(x) = -2 + \sum_{n=1}^{\infty}(-1)^{n+1}(x-2)^n.$

(iii) $f(x) = \sum_{n=0}^{\infty}(-1)^n(x-1)^n.$

Q2

(i) $f(x) = e^a \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}.$

(ii) $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$

(iii) $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = 2.718055556.$

Q3 $f(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$

Q4

(i) $f(x) = \sqrt{x+1} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$.

(ii) $\sqrt{2} = f(1) = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} = 1.3984375$.