University of Chester

Maths Notes for first year Engineers

1. Matrices

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Contents

Chapter 1

Matrices

1.1 What is a matrix?

Definition 1.1 *A Matrix is a rectangular array of numbers.*

Example 1.2 $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ *is a matrix.*

Example 1.3 $\begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$ *is a matrix.*

Example 1.4 $\begin{pmatrix} 2 & 1 & 2 \\ 3 & 5 & 2 \end{pmatrix}$ *is a matrix.*

Note : A Matrix is defined by the number of rows and columns that it contains. A $n \times m$ matrix is a matrix of n rows and m columns.

Example 1.5 *Example 1.2 is a* 2×2 *matrix, example 1.3 is a* 3×3 *matrix and example 1.4 is a* 2×3 *matrix.*

Exercises 1.1

Q1 What types of matrices are the following :

(i) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 3 & 1 \\ 5 & 1 \\ 2 & 3 \end{pmatrix}$ (iv) $\begin{pmatrix} 3 & 3 & 1 & 0 \\ 0 & 5 & 1 & 0 \\ 5 & 1 & -1 & 7 \end{pmatrix}$

(v) $\begin{pmatrix} 3 & 3 & 0 \\ 1 & 1 & -1 \\ 1 & -5 & 2 \\ 5 & 3 & 1 \end{pmatrix}$.

1.2 Addition and Subtraction of matrices

We can only add or subtract matrices of the same type. When two matrices are of the same type, we add/subtract componentwise.

Example 1.6 *Let* $A = \begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 5 \\ 3 & -5 \end{pmatrix}$, *then*

$$A + B = \begin{pmatrix} 2+1 & 2+5 \\ 5+3 & 5+(-5) \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 8 & 0 \end{pmatrix}.$$

Example 1.7 *Let* $A = \begin{pmatrix} 2 & 5 & 3 \\ -1 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 1 & 5 \\ 2 & -1 & 3 \\ 7 & 8 & 9 \end{pmatrix}$, *then*

$$A - B = \begin{pmatrix} 2-1 & 5-1 & 3-5 \\ -1-2 & 0-(-1) & 1-3 \\ 3-7 & 2-8 & 1-9 \end{pmatrix} = \begin{pmatrix} 1 & 4 & -2 \\ -3 & 1 & -2 \\ -4 & -6 & -8 \end{pmatrix}.$$

Exercises 1.2

Q1 Let $A = \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 3 \\ -1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 3 & 2 \end{pmatrix}$. Calculate where possible :

(*i*) $A + B$.

(*ii*) $A - B$.

(*iii*) $B - C$.

(*iv*) $B + A$.

(*v*) $D + A$.

(*vi*) $A - C$.

Q2 Let $A = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 5 & 2 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 6 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$. Calculate :

(*i*) $A + B$.

(*ii*) $A - C$.

(*iii*) $B + A$.

1.3 Scalar Multiplication

Let A be an $m \times n$ matrix and $\lambda \in \mathbb{R}$ (i.e. λ is a scalar). When we are multiplying a matrix by scalar (i.e. a real number), we simply multiply every entry in the matrix by the scalar.

Example 1.8 *Let* $A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$ *and* $\lambda = 2$, *then*

$$\lambda A = 2A = 2\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 & 2\cdot 2 \\ 2\cdot(-3) & 2\cdot 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -6 & 8 \end{pmatrix}.$$

Example 1.9 *Let* $A = \begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 5 \\ 3 & -5 \end{pmatrix}$, *then*

$$\begin{aligned} 5A + 6B &= \begin{pmatrix} 10 & 10 \\ 25 & 25 \end{pmatrix} + \begin{pmatrix} 6 & 30 \\ 18 & -30 \end{pmatrix} \\ &= \begin{pmatrix} 16 & 40 \\ 43 & -5 \end{pmatrix}. \end{aligned}$$

Exercises 1.3 Q1 Let $A = \begin{pmatrix} 2 & -1 \\ 0 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 3 & -2 \end{pmatrix}$. Find (i) $3A + 5B$, (ii) $A - 3B$, (iii) $2A - \frac{1}{2}B$.

1.4 Zero Matrix

Definition 1.10 *The* ***Zero Matrix*** $(O_{m,n})$ *is an* $m \times n$ *Matrix where each of the entries is zero.*

Example 1.11 $O_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $O_{3,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $O_{4,4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

The zero matrix has an important property. Let A be any $m \times n$ matrix, then

$$\boxed{A + O_{m,n} = A = O_{m,n} + A}.$$

1.5 Multiplication of Matrices

1.5.1 Conformable Matrices

When multiplying matrices, we have to be very careful since we can only multiply matrices that are conformable.

Definition 1.12 *Let A and B be matrices. Then A and B are* ***conformable*** *if the number of columns in A are the same as the number of rows in B.*

Note : If two matrices A and B are conformable, this doesn't necessarily mean that B and A are conformable.

Example 1.13 *$A = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 2 \\ -1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \end{pmatrix}$. A is 2×3 matrix, B is 2×2 matrix , C is 2×2 matrix and D is 3×3 matrix. For the matrices A, B, C and D, Which pairs of matrices are conformable?*

- A and B are not conformable, however B and A are conformable.
- A and C are not conformable, however C and A are conformable.
- A and D are conformable, however D and A are not conformable.
- B and C are conformable, also C and B are conformable.
- B and D are not conformable, also D and B are not conformable.
- C and D are not conformable, also D and C are not conformable.

Exercises 1.5.1

Q1 $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 5 \\ 5 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 5 \\ 2 & 1 \\ 0 & 2 \end{pmatrix}$, $E = \begin{pmatrix} 2 & 1 & 0 \\ 2 & -1 & 9 \\ 0 & 3 & 2 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix}$. For the matrices A, B, C, D, E and F, Which pairs of matrices are conformable?

1.5.2 Multiplying Matrices

Let A be a $n \times p$ matrix and B be a $p \times m$ matrix. Clearly A and B are conformable. Thus when we multiply the matrix A by the matrix B (AB), the resulting matrix is a $n \times m$ matrix. To find the $(i, j)^{\text{th}}$-entry of AB, we single out row i from matrix A and column j from matrix B. Multiply the corresponding entries from the row and columns and add the resulting products.

Example 1.14 *Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 5 \\ -5 & 4 \end{pmatrix}$. Calculate AB.*

Clearly AB is a 2×2 matrix. Thus we have 4 entries in AB, one in (1^{st} row, 1^{st} column), one in (1^{st} row, 2^{nd} column), one in (2^{nd} row, 1^{st} column) and one in (2^{nd} row, 2^{nd} column).

(1^{st} row, 1^{st} column)

Take the 1^{st} row in A and the 1^{st} column in B and multiply corresponding entries and add these together. i.e. $(2)(2) + (2)(-5) = 4 - 10 = -6$.

(1^{st} row, 2^{nd} column)

Take the 1^{st} row in A and the 2^{nd} column in B and multiply corresponding entries and add these together. i.e. $(2)(5)+(2)(4)=10+8=18$.

$\underline{(2^{\text{nd}}\text{ row}, 1^{\text{st}}\text{ column})}$

Take the 2^{nd} row in A and the 1^{st} column in B and multiply corresponding entries and add these together. i.e. $(1)(2)+(3)(-5)=2-15=-13$.

$\underline{(2^{\text{nd}}\text{ row}, 2^{\text{nd}}\text{ column})}$

Take the 2^{nd} row in A and the 2^{nd} column in B and multiply corresponding entries and add these together. i.e. $(1)(5)+(3)(4)=5+12=17$.

$$\therefore AB=\begin{pmatrix}-6 & 18\\ -13 & 17\end{pmatrix}.$$

Note that $BA=\begin{pmatrix}(2)(2)+(5)(1) & (2)(2)+(5)(3)\\ (-5)(2)+(4)(1) & (-5)(2)+(4)(3)\end{pmatrix}=\begin{pmatrix}9 & 19\\ -6 & 2\end{pmatrix}\neq AB$. Thus if A and B are conformable and B and A are conformable, it is not necessarily true that $AB=BA$.

Example 1.15 *Let* $A=\begin{pmatrix}2 & 5\\ 3 & 1\end{pmatrix}$, $B=\begin{pmatrix}3 & 5 & 7\\ -1 & 3 & 1\end{pmatrix}$. *Calculate* AB.

$$\begin{aligned}AB&=\begin{pmatrix}(2)(3)+(5)(-1) & (2)(5)+(5)(3) & (2)(7)+(5)(1)\\ (3)(3)+(1)(-1) & (3)(5)+(1)(3) & (3)(7)+(1)(1)\end{pmatrix}\\ &=\begin{pmatrix}6-5 & 10+15 & 14+5\\ 9-1 & 15+3 & 21+1\end{pmatrix}\\ &=\begin{pmatrix}1 & 25 & 19\\ 8 & 18 & 22\end{pmatrix}\end{aligned}$$

Example 1.16 *Let* $A=\begin{pmatrix}2 & 5\\ 1 & 1\end{pmatrix}$, $B=\begin{pmatrix}3\\ -2\end{pmatrix}$. *Calculate* AB.

$$\begin{aligned}AB&=\begin{pmatrix}(2)(3)+(5)(-2)\\ (1)(3)+(1)(-2)\end{pmatrix}\\ &=\begin{pmatrix}6-10\\ 3-2\end{pmatrix}\\ &=\begin{pmatrix}-4\\ 1\end{pmatrix}.\end{aligned}$$

Example 1.17 *Let* $A=\begin{pmatrix}1 & 5 & 1\\ 1 & -1 & 7\\ 9 & 3 & 2\end{pmatrix}$, $B=\begin{pmatrix}5 & 2 & 3\\ 1 & 2 & 3\\ 1 & 1 & -1\end{pmatrix}$. *Calculate* AB.

$$
\begin{aligned}
AB &= \begin{pmatrix} (1)(5)+(5)(1)+(1)(1) & (1)(2)+(5)(2)+(1)(1) & (1)(3)+(5)(3)+(1)(-1) \\ (1)(5)+(-1)(1)+(7)(1) & (1)(2)+(-1)(2)+(7)(1) & (1)(3)+(-1)(3)+(7)(-1) \\ (9)(5)+(3)(1)+(2)(1) & (9)(2)+(3)(2)+(2)(1) & (9)(3)+(3)(3)+(2)(-1) \end{pmatrix} \\
&= \begin{pmatrix} 5+5+1 & 2+10+1 & 3+15-1 \\ 5-1+7 & 2-2+7 & 3-3-7 \\ 45+3+2 & 18+6+2 & 27+9-2 \end{pmatrix} \\
&= \begin{pmatrix} 11 & 13 & 17 \\ 11 & 7 & -7 \\ 50 & 26 & 34 \end{pmatrix}.
\end{aligned}
$$

Example 1.18 *Let* $A = \begin{pmatrix} 1 & 5 & 1 \\ 1 & -1 & 7 \\ 9 & 3 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$. *Calculate* AB.

$$
\begin{aligned}
AB &= \begin{pmatrix} (1)(5)+(5)(1)+(1)(1) & (1)(2)+(5)(2)+(1)(1) \\ (1)(5)+(-1)(1)+(7)(1) & (1)(2)+(-1)(2)+(7)(1) \\ (9)(5)+(3)(1)+(-2)(1) & (9)(2)+(3)(2)+(-2)(1) \end{pmatrix} \\
&= \begin{pmatrix} 5+5+1 & 2+10+1 & \\ 5-1+7 & 2-2+7 & \\ 45+3-2 & 18+6-2 & 27+9-2 \end{pmatrix} \\
&= \begin{pmatrix} 11 & 13 \\ 11 & 7 \\ 46 & 22 \end{pmatrix}.
\end{aligned}
$$

Exercises 1.5.2

Q1 Let $A = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 7 \\ -2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} -2 & -1 \\ 5 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 4 & 3 \\ -6 & 2 & -7 \end{pmatrix}$, $E = \begin{pmatrix} 3 & 4 \\ 2 & -7 \\ 8 & 10 \end{pmatrix}$, $F = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -7 & 5 \\ 8 & 10 & 1 \end{pmatrix}$, $G = \begin{pmatrix} 5 & 0 & 1 \\ 3 & -2 & 3 \\ 1 & 2 & -4 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -7 & 1 & 1 \\ 8 & 1 & 1 & -1 \end{pmatrix}$.

(a) For the matrices $\{A, B, C, D, E, F, G, H\}$, decide which pairs are conformable.

(b) For each of the pairs of matrices that are conformable, calculate their product.

1.6 Transpose of a Matrix

Definition 1.19 *The transpose of a matrix* A *denoted by* A^T *is obtained by converting the rows of* A *into columns one at a time in sequence.*

Example 1.20 *Let* $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -7 & 1 \\ 8 & 1 & 1 \end{pmatrix}$, *then* $A^T = \begin{pmatrix} 1 & 2 & 8 \\ 0 & -7 & 1 \\ 2 & 1 & 1 \end{pmatrix}$. *Also* $(A^T)^T = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -7 & 1 \\ 8 & 1 & 1 \end{pmatrix} = A$.

Example 1.21 *Let* $A = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -1 \\ 3 & -5 \end{pmatrix}$, *then*

$$(AB)^T = \left[\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & -5 \end{pmatrix}\right]^T = \left[\begin{pmatrix} 3-3 & -3+5 \\ 2+12 & -2-20 \end{pmatrix}\right]^T \left[\begin{pmatrix} 0 & 2 \\ 14 & -22 \end{pmatrix}\right]^T = \begin{pmatrix} 0 & 14 \\ 2 & -22 \end{pmatrix}.$$

Also

$$B^T A^T = \begin{pmatrix} 1 & 3 \\ -1 & -5 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 3-3 & 2+12 \\ -3+5 & -2-22 \end{pmatrix} = \begin{pmatrix} 0 & 14 \\ 2 & -22 \end{pmatrix}.$$

In general, we have that

- $(A^T)^T = A$ for any $m \times n$ matrix A.
- $(AB)^T = B^T A^T$ for any two conformable matrices A and B.

Exercises 1.6

Q1 Let $A = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 7 \\ -2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} -2 & -1 \\ 5 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 4 & 3 \\ -6 & 2 & -7 \end{pmatrix}$, $E = \begin{pmatrix} 3 & 4 \\ 2 & -7 \\ 8 & 10 \end{pmatrix}$, $F = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -7 & 5 \\ 8 & 10 & 1 \end{pmatrix}$, $G = \begin{pmatrix} 5 & 0 & 1 \\ 3 & -2 & 3 \\ 1 & 2 & -4 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -7 & 1 & 1 \\ 8 & 1 & 1 & -1 \end{pmatrix}$.

(a) For the matrices $\{A, B, C, D, E, F, G, H\}$, find the transpose of each matrix.

(b) Show $(BC)^T = C^T B^T$.

(c) Show $(FG)^T = G^T F^T$.

1.7 Identity Matrix

Definition 1.22 *The* ***Identity Matrix*** (I_n) *is an* $n \times n$ *Matrix where each diagonal entry is 1 and each off diagonal entry is 0.*

Example 1.23 $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

The Identity matrix has a very important property. Let A be any $n \times n$ matrix, then

$$\boxed{A \cdot I_n = A = I_n \cdot A}.$$

1.8 The Inverse of a 2×2 matrix

1.8.1 The Determinant and adjoint matrix of a 2×2 matrix

The determinant of a matrix is denoted by $|A|$ and the adjoint matrix by A^*. Let $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$, then $|A| = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$ and $A^* = \begin{pmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{pmatrix}$.

Example 1.24 *Let* $A = \begin{pmatrix} 1 & 5 \\ -7 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 5 & 1 \\ 3 & -5 \end{pmatrix}$. *Investigate if (i)* $|AB| = |A||B|$ *(ii)* $(A+B)^* = A^* + B^*$ *(iii)* $(AB)^* = B^*A^*$.

(i) $|A| = (1)(1) - (5)(-7) = 1 + 35 = 36$. $|B| = (5)(-5) - (3)(1) = -25 - 3 = -28$ *and* $|A||B| = (36)(-28) = -1008$.

$AB = \begin{pmatrix} 1 & 5 \\ -7 & 1 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 3 & -5 \end{pmatrix} = \begin{pmatrix} 5+15 & 1-25 \\ -35+3 & -7-5 \end{pmatrix} = \begin{pmatrix} 20 & -24 \\ -32 & -12 \end{pmatrix}$. *Thus* $|AB| = (20)(-12) - (-32)(-24) = -1008$. *Therefore* $\boxed{|AB| = |A||B|}$.

(ii) $A + B = \begin{pmatrix} 6 & 6 \\ -4 & -4 \end{pmatrix}$, $\therefore (A+B)^* = \begin{pmatrix} -4 & -6 \\ 4 & 6 \end{pmatrix}$.

$A^* = \begin{pmatrix} 1 & -5 \\ 7 & 1 \end{pmatrix}$, $B^* = \begin{pmatrix} -5 & -1 \\ -3 & 5 \end{pmatrix}$. $\therefore A^* + B^* = \begin{pmatrix} -4 & -6 \\ 4 & 6 \end{pmatrix}$. *Therefore* $\boxed{(A+B)^* = A^* + B^*}$.

(iii) $AB = \begin{pmatrix} 20 & -24 \\ -32 & -12 \end{pmatrix}$. $\therefore (AB)^* = \begin{pmatrix} -12 & 24 \\ 32 & 20 \end{pmatrix}$.

$B^*A^* = \begin{pmatrix} -5 & -1 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 1 & -5 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} -5-7 & 25-1 \\ -3+35 & 15+5 \end{pmatrix} = \begin{pmatrix} -12 & 24 \\ 32 & 20 \end{pmatrix}$. *Therefore* $\boxed{(AB)^* = B^*A^*}$.

In general, we have that $|AB| = |A||B|$, for any two $n \times n$ matrices A and B.

Exercises 1.8.1

Q1 Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -3 \\ 3 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Show that (i) $|AB| = |A||B|$ (ii) $|BC| = |B||C|$ (iii) $(A+C)^* = A^* + C^*$ (iv) $(A+B)^* = A^* + B^*$ (v) $(AC)^* = C^*A^*$ (vi) $(BC)^* = C^*B^*$ (vii) $(A^*)^* = A$ $(viii)$ $(B^*)^* = B$.

1.8.2 The inverse of a 2×2 matrix

Definition 1.25 *A square Matrix is a matrix where the number of rows and columns are equal.*

We only try to invert square matrices. In general not every square matrix is invertible. If a square matrix has an inverse, it is said to be *invertible* or $non-singular$ otherwise it is said to be $singular$. If a matrix is invertible, we can calculate it's inverse (denoted by A^{-1}).

In general the formula for the inverse of an $n \times n$ matrix is :

$$\boxed{A^{-1} = \frac{1}{|A|}.A^*}.$$

where $|A|$ is the determinant of A and A^* is the adjoint matrix of A. Therefore the inverse of a 2×2 matrix $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$ is

$$A^{-1} = \frac{1}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}} \begin{pmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{pmatrix}.$$

Thus if $|A| = 0$, A is *singular*. Also if $|A| \neq 0$, A is *non − singular*.

Example 1.26 *Let* $A = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$. *Find* A^{-1}.
$|A| = 10 - 12 = -2$. *Also* $A^* = \begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix}$. *Therefore* $A^{-1} = \frac{1}{-2}\begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix}$.

Example 1.27 *Let* $A = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$. *Find* $|A|$. *Does* A^{-1} *exist?*
$|A| = (3)(4) - (2)(6) = 12 - 12 = 0$. *Therefore* A^{-1} *does not exist.*

In general inverse matrix has a very important property:

$$\boxed{A \cdot A^{-1} = I_n = A^{-1} \cdot A}.$$

where A is a non-singular $n \times n$ matrix.
Exercises 1.8.2

Q1 Let $A = \begin{pmatrix} 7 & 3 \\ 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 7 & \frac{1}{3} \\ 63 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 5 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$. For the matrices $\{A, B, C, D\}$, calculate where possible their inverses.

1.9 The inverse of a 3×3 matrix

In this section we will formulate the determinant, adjoint matrix and the inverse of a 3×3 matrix.

1.9.1 The Determinant and adjoint matrix of a 3×3 matrix

Let $A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$, then

$$\begin{aligned} |A| &= a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3}\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} \\ &= a_{1,1}[a_{2,2}a_{3,3} - a_{2,3}a_{3,2}] - a_{1,2}[a_{2,1}a_{3,3} - a_{2,3}a_{3,1}] + a_{1,3}[a_{2,1}a_{3,2} - a_{2,2}a_{3,1}] \end{aligned}$$

Example 1.28 *Let* $A = \begin{pmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix}$. *Find* $|A|$.

$$\begin{aligned} |A| &= (-2)\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} - (1)\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} + (1)\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ &= (-2)[(1)(2)-(3)(5)] - (1)[(2)(2)-(5)(1)] + (1)[(2)(3)-(1)(1)] \\ &= 26 + 1 + 5 \\ &= 32. \end{aligned}$$

1.9.2 The Matrix of Cofactors and the Adjoint matrix

For every $n \times n$ *matrix* A*, there exists a matrix of Cofactors* C*. So for each entry in* A $(a_{i,j})$*, there exists an* $n \times n$ *corresponding cofactor* $c_{i,j}$*. We calculate* $c_{i,j}$ *using the formula*

$$c_{i,j} = (-1)^{i+j} M_{i,j}$$

where $M_{i,j}$ *is the determinant of the submatrix obtain by eliminating the* i*-th row and the* j*-th column from* A*.*

Example 1.29 *Let* $A = \begin{pmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix}$. *Then the matrix of Cofactors is a* 3×3 *matrix i.e.* $C = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}$. *We use the* $c_{i,j} = (-1)^{i+j} M_{i,j}$ *to calculate* $c_{1,1}, c_{1,2}, c_{1,3}, \ldots, c_{3,3}$.

$$\boxed{c_{1,1}}$$

We simply replace i *with* 1 *and* j *with* 1 *in the* $c_{i,j}$ *formula. Therefore* $c_{11} = (-1)^{1+1} M_{11}$*. Now* M_{11} *is the determinant of the submatrix obtained by eliminating row* 1 *and column* 1 *in* A*.*

$$M_{11} = \begin{vmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} = (1)(2) - (5)(3) = 2 - 15 = -13.$$

Therefore $c_{11} = (-1)^2(-13) = -13$.

$$\boxed{c_{1,2}}$$

$$c_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = -[(2)(2) - (5)(1)] = -(4-5) = 1.$$

$$\boxed{c_{1,3}}$$

$$c_{13} = (-1)^{1+3}M_{13} = +\begin{vmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{vmatrix} = +\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = +[(2)(3)-(1)(1)] = 6-1 = 5.$$

$\boxed{c_{2,1}}$

$$c_{21} = (-1)^{2+1}M_{21} = -\begin{vmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -[(1)(2)-(1)(3)] = -(2-3) = 1.$$

$\boxed{c_{2,2}}$

$$c_{22} = (-1)^{2+2}M_{22} = +\begin{vmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{vmatrix} = +\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} = (-2)(2)-(1)(1) = -4-1 = -5.$$

$\boxed{c_{2,3}}$

$$c_{23} = (-1)^{2+3}M_{23} = -\begin{vmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} = -[(-2)(3)-(1)(1)] = -(-6-1) = 7.$$

$\boxed{c_{3,1}}$

$$c_{31} = (-1)^{3+1}M_{31} = +\begin{vmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{vmatrix} = +\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = (1)(5)-(1)(1) = 5-1 = 4.$$

$\boxed{c_{3,2}}$

$$c_{32} = (-1)^{3+2}M_{32} = -\begin{vmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} -2 & 1 \\ 2 & 5 \end{vmatrix} = -[(-2)(5)-(1)(2)] = -(-10-2) = 12.$$

$\boxed{c_{3,3}}$

$$c_{33} = (-1)^{3+3}M_{33} = +\begin{vmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{vmatrix} = +\begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} = (-2)(1)-(1)(2) = -2-2 = -4.$$

Therefore $C = \begin{pmatrix} -13 & 1 & 5 \\ 1 & -5 & 7 \\ 4 & 12 & -4 \end{pmatrix}.$

In general it is well known that:

Theorem 1.30 *The Adjoint Matrix of a matrix is equal to the transpose of the corresponding matrix of cofactors, i.e.* $\boxed{A^* = C^T}$.

Example 1.31 *Let* $A = \begin{pmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix}$. *Then*

$$A^* = C^T = \begin{pmatrix} -13 & 1 & 5 \\ 1 & -5 & 7 \\ 4 & 12 & -4 \end{pmatrix}^T = \begin{pmatrix} -13 & 1 & 4 \\ 1 & -5 & 12 \\ 5 & 7 & -4 \end{pmatrix}.$$

Consider AA^*.

$$\begin{aligned} AA^* &= \begin{pmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} -13 & 1 & 4 \\ 1 & -5 & 12 \\ 5 & 7 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 26+1+5 & -2-5+7 & -8+12-4 \\ -26+1+25 & 2-5+35 & 8+12-20 \\ -13+3+10 & 1-15+14 & 4+36-8 \end{pmatrix} \\ &= \begin{pmatrix} 32 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{pmatrix} \\ &= \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix} \end{aligned}.$$

It can also easily shown (for this example that $A^*A = |A|.I_3$. *In general we have that:*

Theorem 1.32

$$AA^* = |A|.I_n = A^*A.$$

1.9.3 The inverse of a 3×3 matrix

Recall that

$$A^{-1} = \frac{1}{|A|} A^*.$$

Example 1.33 *Let* $A = \begin{pmatrix} -2 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix}$, *recall that* $|A| = 32$ *and* $A^* = \begin{pmatrix} -35 & 12 & 5 \\ -1 & -4 & 11 \\ 21 & 8 & -3 \end{pmatrix}$. *Therefore*

$$A^{-1} = \frac{1}{32} \begin{pmatrix} -13 & 1 & 4 \\ 1 & -5 & 12 \\ 5 & 7 & -4 \end{pmatrix} = \begin{pmatrix} \frac{-13}{32} & \frac{1}{32} & \frac{4}{32} \\ \frac{1}{32} & \frac{-5}{32} & \frac{12}{32} \\ \frac{5}{32} & \frac{7}{32} & \frac{-4}{32} \end{pmatrix}.$$

Example 1.34 *Let* $A = \begin{pmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{pmatrix}$. *Find* $|A|$. *Does* A^{-1} *exist ? Explain ?*

$$\begin{aligned}
|A| &= (-2)\begin{vmatrix} 1 & -2 \\ 8 & 4 \end{vmatrix} - (7)\begin{vmatrix} 5 & -2 \\ 3 & 4 \end{vmatrix} + (6)\begin{vmatrix} 5 & 1 \\ 3 & 8 \end{vmatrix} \\
&= (-2)[(1)(4) - (-2)(8)] - (7)[(5)(4) - (-2)(3)] + (6)[(5)(8) - (3)(1)] \\
&= (-2)[20] - (7)[26] + (6)[37] \\
&= -40 - 182 + 222 \\
&= 0.
\end{aligned}$$

$|A| = 0$, *therefore* A^{-1} *doesn't exist.*

Exercises 1.9.2

Q1 Let $A = \begin{pmatrix} 2 & 3 & 4 \\ -5 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 1 & 2 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{pmatrix}$.

(i) Find $|A|$, $|B|$ and $|C|$.

(ii) Calculate AA^*, B^*B and CC^*.

(iii) For the matrices $\{A, B, C\}$, calculate their inverses where possible.

1.10 Solving equations involving Matrices

If I gave you the linear equation $ax + b = 0$ and asked you to solve it, you would say that $x = -\frac{b}{a}$. However if I asked you to solve $AX = B$ where A, X and B are matrices then you cannot say that $X = -\frac{B}{A}$ because we cannot divide matrices. Therefore we must solve equations involving matrices a different way.

1.10.1 Solving equations involving Matrices

Example 1.35 *Let* $A = \begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 7 & 1 \\ -1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *and* X *be a matrix. Solve the following equations involving matrices for* X*:*

(i) $AX = B$

(ii) $XB = C$

(iii) $BX = D$

(iv) $ABX = C$

(v) $BXA = C$

(vi) $XA = B + 2C$.

$\boxed{(i) \quad AX = B}$.

We first pre-multiply (multiply on left) by A^{-1} on both sides of the equation.

$$A^{-1}AX = A^{-1}B$$

At this point, we know from the property of A^{-1} that $A^{-1}A = I_2$.

$$I_2X = A^{-1}B$$

Also we know that $I_2X = X$ from the property of I_2.

$$\therefore X = A^{-1}B$$

We are now ready to calculate X.

$$X = A^{-1}B = \frac{1}{13}\begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 7 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 23 & 3 \\ -12 & -1 \end{pmatrix} = \begin{pmatrix} \frac{23}{13} & \frac{3}{13} \\ \frac{-12}{13} & \frac{-1}{13} \end{pmatrix}.$$

$\boxed{(ii) \quad XB = C}$.

$$\begin{aligned} XB &= C \\ XBB^{-1} &= CB^{-1} && \text{post-multiply by } B^{-1} \\ XI_2 &= CB^{-1} && BB^{-1} = I_2 \\ X &= CB^{-1} && XI_2 = X \end{aligned}$$

$$X = CB^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\frac{1}{1}\begin{pmatrix} 0 & -1 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 1 & 8 \end{pmatrix}.$$

$\boxed{(iii) \quad BX = D}$.

$$\begin{aligned} BX &= D \\ B^{-1}BX &= B^{-1}D && \text{pre-multiply by } B^{-1} \\ I_2X &= B^{-1}D && BB^{-1} = I_2 \\ X &= B^{-1}D && I_2X = X \end{aligned}$$

$$X = B^{-1}D = \begin{pmatrix} 0 & -1 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}.$$

$\boxed{(iv) \quad ABX = C}$.

$$\begin{aligned} ABX &= C \\ A^{-1}ABX &= A^{-1}C \quad \text{pre-multiply by } A^{-1} \\ I_2BX &= B^{-1}C \quad AA^{-1} = I_2 \\ BX &= A^{-1}D \quad I_2B = B \\ B^{-1}BX &= B^{-1}A^{-1}C \quad \text{pre-multiply by } B^{-1} \\ I_2X &= B^{-1}A^{-1}C \quad BB^{-1} = I_2 \\ X &= B^{-1}A^{-1}C \quad I_2X = X \end{aligned}$$

$$X = B^{-1}A^{-1}C = \begin{pmatrix} 0 & -1 \\ 1 & 7 \end{pmatrix} \frac{1}{13} \begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 1 & -5 \\ -2 & 33 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$
$$= \frac{1}{13} \begin{pmatrix} 5 & -4 \\ -33 & 29 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} & \frac{-4}{13} \\ \frac{-33}{13} & \frac{29}{13} \end{pmatrix}.$$

$\boxed{(v) \quad BXA = C}$.

$$\begin{aligned} BXA &= C \\ B^{-1}BXA &= B^{-1}C \quad \text{pre-multiply by } B^{-1} \\ I_2XA &= B^{-1}C \\ XA &= B^{-1}C \\ XAA^{-1} &= B^{-1}CA^{-1} \quad \text{post-multiply by } A^{-1} \\ XI_2 &= B^{-1}CA^{-1} \\ X &= B^{-1}CA^{-1} \end{aligned}$$

$$X = B^{-1}CA^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \frac{1}{13} \begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 1 & -1 \\ -7 & 8 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix}$$
$$= \frac{1}{13} \begin{pmatrix} 4 & -7 \\ -29 & 54 \end{pmatrix} = \begin{pmatrix} \frac{4}{13} & \frac{-7}{13} \\ \frac{-29}{13} & \frac{54}{13} \end{pmatrix}.$$

$\boxed{(vi) \quad XA = B + 2C}$.

$$\begin{aligned} XA &= B + 2C \\ XAA^{-1} &= (B + 2C)A^{-1} \quad \text{post-multiply by } A^{-1} \\ XI_2 &= (B + 2C)A^{-1} \\ X &= (B + 2C)A^{-1} \end{aligned}$$

$$X = (B + 2C)A^{-1} = \left[\begin{pmatrix} 7 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -2 & 2 \end{pmatrix} \right] \frac{1}{13} \begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 7 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix}$$
$$= \frac{1}{13} \begin{pmatrix} 18 & 1 \\ -11 & 16 \end{pmatrix} = \begin{pmatrix} \frac{18}{13} & \frac{1}{13} \\ \frac{-11}{13} & \frac{16}{13} \end{pmatrix}.$$

Example 1.36 *Let* $A = \begin{pmatrix} -1 & 1 & 2 \\ 3 & 0 & 5 \\ 1 & 7 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. *Solve* $AX = B$ *for* X *where* X *is a matrix.*

$\boxed{AX = B}$.

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ I_3X &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

$$X = A^{-1}B = \frac{1}{32}\begin{pmatrix} -13 & 1 & 4 \\ 1 & -5 & 12 \\ 5 & 7 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{32}\begin{pmatrix} -17 \\ -11 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{-17}{32} \\ \frac{-11}{32} \\ \frac{9}{32} \end{pmatrix}.$$

Exercises 1.10.1

Q1 Let $A = \begin{pmatrix} 7 & 3 \\ 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 5 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $E = \begin{pmatrix} 2 & 3 & 4 \\ -5 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $F = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$
and X be a matrix. Solve the following equations involving matrices for X:

(*i*) $AX = B$

(*ii*) $AX = D$

(*iii*) $BX = D$

(*iv*) $XAB = C$

(*v*) $AXB = C$

(*vi*) $AX = B + 2C$

(*vii*) $EX = F$.

1.10.2 Solving A System Of Equations Using Matrices

Example 1.37 *Solve the following simultaneous equations*

$$\begin{aligned} x + y &= 1 \\ x - y &= 7 \end{aligned}$$

We can rewrite this system of equations as an equation with matrices as follows:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

where $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is a 2×2 matrix consisting of the coefficients of the x's and y's on the left hand side, $\begin{pmatrix} x \\ y \end{pmatrix}$ is a 2×1 matrix consisting of what we are solving for and $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$ is a 2×1 matrix consisting of the numbers on the right hand side. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ and we have $AX = B$. So solving this system of equations is equivalent to solving a matrix equation where we solving for the matrix X.

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ I_2X &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

$X = \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}B = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -8 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$. Therefore $x = 4$ and $y = -3$.

Example 1.38 *Solve the following simultaneous equations*

$$\begin{aligned} 3x - y &= 11 \\ 3x - 2y &= 13 \end{aligned}$$

$$\begin{pmatrix} 3 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix} \Longrightarrow AX = B$$

where $A = \begin{pmatrix} 3 & -1 \\ 3 & -2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$.

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ I_2X &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

$X = \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}B = \frac{1}{-3} \begin{pmatrix} -2 & 1 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 11 \\ 13 \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} -9 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Therefore $x = 3$ and $y = -2$.

Example 1.39 *Solve the following equations*

$$\begin{aligned} 2x + y + z &= 8 \\ 5x - 3y + 2z &= 3 \\ 7x + y + 3z &= 20 \end{aligned}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 5 & -3 & 2 \\ 7 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ 20 \end{pmatrix} \Longrightarrow AX = B$$

where $A = \begin{pmatrix} 2 & 1 & 1 \\ 5 & -3 & 2 \\ 7 & 1 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 8 \\ 3 \\ 20 \end{pmatrix}$.

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ I_3X &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B = \dfrac{1}{3}\begin{pmatrix} -11 & -2 & 5 \\ -1 & -1 & 1 \\ 26 & 5 & -11 \end{pmatrix} \begin{pmatrix} 8 \\ 3 \\ 20 \end{pmatrix} = \dfrac{1}{3}\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$. Therefore $x = 2$ and $y = 3$ and $z = 1$.

Exercises 1.10.2

Q1 Solve the following systems of equations using matrix methods.

(*i*)

$$\begin{aligned} 3x + y &= 7 \\ 2x - 2y &= 10 \end{aligned}$$

(*ii*)

$$\begin{aligned} 2x - y - 2z &= 9 \\ 3y + 4z &= -11 \\ x - z &= 4 \end{aligned}$$

1.11 Answers

Exercises 1.1

Q1 *(i)* 2×1 matrix, *(ii)* 1×2 matrix, *(iii)* 3×2 matrix, *(iv)* 3×4 matrix, *(v)* 4×3 matrix.

Exercises 1.2

Q1 *(i)* $\begin{pmatrix} 7 & 8 \\ 2 & 3 \end{pmatrix}$, *(ii)* $\begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$, *(iii)* $\begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}$, *(iv)* $\begin{pmatrix} 7 & 8 \\ 2 & 3 \end{pmatrix}$, *(v)* not possible, *(vi)* $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Q2 *(i)* $\begin{pmatrix} 5 & 10 & 5 \\ 6 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, *(ii)* $\begin{pmatrix} -4 & 5 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$, *(iii)* $\begin{pmatrix} 5 & 10 & 5 \\ 6 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

Exercises 1.3

Q1 *(i)* $\begin{pmatrix} 11 & 17 \\ 15 & -19 \end{pmatrix}$, *(ii)* $\begin{pmatrix} -1 & -13 \\ -9 & 3 \end{pmatrix}$, *(iii)* $\begin{pmatrix} 7/2 & -4 \\ -3/2 & -5 \end{pmatrix}$.

Exercises 1.5.1

Q1 Conformable are the pairs: A and B, A and C, B and A, B and C, C and D, C and E, C and F, D and A, D and B, D and C, E and D, E and F, F and D, F and E.

Exercises 1.5.2

Q1 (a) Conformable are the pairs: B and A, B and C, B and D, C and A, C and B, C and D, D and E, D and F, D and G, D and H, E and A, E and B, E and C, E and D, F and E, F and G, F and E, G and F, G and H.

(b) $BA = \begin{pmatrix} -18 \\ -15 \end{pmatrix}$, $BC = \begin{pmatrix} 33 & 20 \\ 19 & 11 \end{pmatrix}$, $BD = \begin{pmatrix} -42 & 18 & -49 \\ -18 & -2 & -27 \end{pmatrix}$, $CA = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$, $CB = \begin{pmatrix} 0 & -17 \\ -1 & 44 \end{pmatrix}$, $CD = \begin{pmatrix} 6 & -10 & 1 \\ -18 & 26 & -6 \end{pmatrix}$, $DE = \begin{pmatrix} 32 & 2 \\ -70 & -108 \end{pmatrix}$, $DF = \begin{pmatrix} 32 & 2 & 23 \\ -58 & -78 & -9 \end{pmatrix}$, $DG = \begin{pmatrix} 17 & -2 & 0 \\ -31 & -18 & -28 \end{pmatrix}$, $DH = \begin{pmatrix} 32 & -25 & 7 & 1 \\ -58 & -21 & -17 & -23 \end{pmatrix}$, $EA = \begin{pmatrix} -3 \\ 27 \\ -6 \end{pmatrix}$, $EB = \begin{pmatrix} -5 & 33 \\ 16 & -7 \\ -12 & 86 \end{pmatrix}$, $EC = \begin{pmatrix} 14 & 9 \\ -39 & -23 \\ 34 & 22 \end{pmatrix}$, $ED = \begin{pmatrix} -24 & 20 & -19 \\ 42 & -6 & 55 \\ -60 & 52 & -46 \end{pmatrix}$, $FE = \begin{pmatrix} 17 & 31 \\ 32 & 107 \\ 52 & -28 \end{pmatrix}$, $FG = \begin{pmatrix} 4 & 4 & -12 \\ -6 & 24 & -39 \\ 81 & -18 & 34 \end{pmatrix}$, $FH = \begin{pmatrix} 15 & 9 & 3 & 0 \\ 38 & 54 & 2 & -6 \\ 36 & -69 & 27 & 33 \end{pmatrix}$, $GE = \begin{pmatrix} 23 & 30 \\ 29 & 56 \\ -25 & -50 \end{pmatrix}$, $GF = \begin{pmatrix} 13 & 5 & 11 \\ 23 & 41 & -1 \\ -27 & -55 & 88 \end{pmatrix}$, $GH = \begin{pmatrix} 13 & 1 & 11 & 14 \\ 23 & 17 & 7 & 4 \\ -27 & -18 & 0 & 9 \end{pmatrix}$.

Exercises 1.6

Q1 (a) $A^T = \begin{pmatrix} 3 & -3 \end{pmatrix}$, $B^T = \begin{pmatrix} 1 & -2 \\ 7 & 3 \end{pmatrix}$, $C^T = \begin{pmatrix} -2 & 5 \\ -1 & 3 \end{pmatrix}$, $D^T = \begin{pmatrix} 0 & -6 \\ 4 & 2 \\ 3 & -7 \end{pmatrix}$, $E^T = \begin{pmatrix} 3 & 2 & 8 \\ 4 & -7 & 10 \end{pmatrix}$, $F^T = \begin{pmatrix} 1 & 2 & 8 \\ -1 & -7 & 10 \\ 2 & 5 & 1 \end{pmatrix}$, $G^T = \begin{pmatrix} 5 & 3 & 1 \\ 0 & -2 & 2 \\ 1 & 3 & -4 \end{pmatrix}$, $H^T = \begin{pmatrix} 1 & 2 & 8 \\ 0 & -7 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix}$.

(b) $(BC)^T = C^T B^T = \begin{pmatrix} 33 & 19 \\ 20 & 11 \end{pmatrix}$.

(c) $(FG)^T = G^T F^T = \begin{pmatrix} 4 & -6 & 81 \\ 4 & 24 & -18 \\ -10 & -39 & 34 \end{pmatrix}$.

Exercises 1.8.1

Q1 (i) $|AB| = |A||B| = 20$, (ii) $|BC| = |B||C| = -8$, (iii) $(A+C)^* = A^* + C^* = \begin{pmatrix} 8 & -5 \\ -4 & 3 \end{pmatrix}$, (iv) $(A+B)^* = A^* + B^* = \begin{pmatrix} 9 & 0 \\ -4 & 1 \end{pmatrix}$, (v) $(AC)^* = C^* A^* = \begin{pmatrix} 10 & 14 \\ -13 & 11 \end{pmatrix}$, (vi) $(BC)^* = C^* B^* = \begin{pmatrix} 26 & 14 \\ -18 & -10 \end{pmatrix}$.

Exercises 1.8.2

Q1 $A^{-1} = \begin{pmatrix} 1 & -3/2 \\ -2 & 7/2 \end{pmatrix}$, B is singular, $C^{-1} = \begin{pmatrix} 0 & 1/5 \\ -1 & 1/5 \end{pmatrix}$, D is singular.

Exercises 1.9.2

Q1 (i) $|A| = -45$, $|B| = 0$, $|C| = -65$.

(ii) $AA^* = -1/45 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B^*B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $CC^* = -1/65 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(iii) $A^{-1} = \begin{pmatrix} 1/ & -1/9 & 2/45 \\ -29/15 & 2/9 & 32/45 \\ 5/3 & -1/9 & -5/9 \end{pmatrix}$, B is singular, $C^{-1} = \begin{pmatrix} -4/5 & -22/65 & 27/65 \\ 1/5 & 8/65 & 2/65 \\ -1/5 & -1/5 & 1/5 \end{pmatrix}$.

Exercises 1.10.1

Q1 (i) $X = A^{-1}B = \begin{pmatrix} -13/2 & -1 \\ -31/2 & 2 \end{pmatrix}$, (ii) $X = A^{-1}D = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, (iii) $X = B^{-1}D = \begin{pmatrix} 2/5 \\ -3/5 \end{pmatrix}$, (iv) $X = CB^{-1}A^{-1} = 1/10 \begin{pmatrix} -4 & 7 \\ -46 & 73 \end{pmatrix}$, (v) $X = A^{-1}CB^{-1} = 1/10 \begin{pmatrix} 45 & -1 \\ -105 & 24 \end{pmatrix}$, (vi) $X = A^{-1}(B+2C) = \begin{pmatrix} -15/2 & -10 \\ 37/2 & 23 \end{pmatrix}$, (vii) $X = E^{-1}F = \begin{pmatrix} -4/45 \\ -109/45 \\ 19/9 \end{pmatrix}$.

Exercises 1.10.2

Q1 (*i*) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, (*ii*) $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.